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Dynamic conditional score models: a review

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Abstract: We provide a review of the recent class of dynamic conditional score (DCS) time series models. For a DCS model, each dynamic equation that drives a time-varying parameter is updated by the conditional score of the log-likelihood with respect to the same time-varying parameter. DCS models can be related to the classical observation-driven and parameter-driven time series models from the body of econometric literature. We explain the main differences between the classical and DCS time series models, by comparing two examples, respectively, the Gaussian signal plus noise model and the dynamic Student's *t* location model. We also present applications of three DCS models for financial and economic data: Firstly, we present the QAR (quasi-autoregressive) plus Beta-*t*-EGARCH (exponential autoregressive conditional heteroscedasticity) model, which is a score-driven expected return plus score-driven volatility model. We use this model for daily return data on the DAX (Deutscher Aktienindex) equity index for the period of 5th January 1988 to 29th December 2017. We compare the QAR plus Beta-*t*-EGARCH model with the classical AR plus *t*-GARCH model. Secondly, we present the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model, which is used for daily AFN/USD (Afghan Afghani/ United States Dollar) currency exchange rate data for the period of 1st March 2007 to 7th July 2017. We compare the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model with a classical local level, classical dynamic seasonality plus GARCH model. Thirdly, we present the QVAR (quasi-vector autoregressive) model, which is a score-driven multivariate dynamic model of location. We use this model for monthly US inflation rate and US unemployment rate data for the period of 1st January 1948 to 1st December 2017. We compare the QVAR model with the classical VAR model. For all applications, we present the estimation results and the model diagnostics of DCS and classical models, which suggest that each DCS model is superior to the classical alternative.

Keywords: dynamic conditional score (DCS) models; quasi-autoregressive (QAR) model; Beta-*t*-EGARCH (exponential generalized autoregressive conditional heteroscedasticity) model; score-driven seasonality; quasi-vector autoregressive (QVAR) model; DAX (Deutscher Aktienindex); AFN/USD (Afghan Afghani / United States Dollar) exchange rate; US inflation and unemployment rates

JEL classification codes: C22; C32; C52

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1. Introduction

In this article, we provide a review of a recent class of score-driven time series models, which is introduced in the works of Creal, Koopman and Lucas (2013) and Harvey (2013). Those authors name their score-driven models as generalized autoregressive score (GAS) and dynamic conditional score (DCS) models, respectively (in the present paper, we use the DCS notation). For a DCS model, each dynamic equation that drives a time-varying model parameter (e.g. mean, variance or correlation coefficient) is updated by the conditional score of the log-likelihood (LL) (hereinafter, score function) with respect to the same time-varying parameter. DCS models can be related to classical 'observation-driven and parameter-driven time series models' (Cox, 1981) from the body of literature. We present those relationships in Section 2.

In Section 3, we explain the main differences between the classical and DCS time series models, by comparing the 'Gaussian signal plus noise model' (Harvey, 1989) and the 'dynamic Student's *t* location model' (Harvey, 2013). The Gaussian signal plus noise model and the dynamic Student's *t* location model are simple examples from each class of models, respectively, which are presented here to highlight the main differences between the classical and DCS models.

In Sections 4 to 6, we present applications of three DCS models for financial and economic time series variables. In Section 4, we present the QAR (quasi-autoregressive) (Harvey, 2013) plus Beta-*t*-EGARCH (exponential autoregressive conditional heteroscedasticity) (Harvey and Chakravarty, 2008; Harvey, 2013) model, which is a score-driven expected return plus score-driven volatility model. We use this model for daily return data on the DAX (Deutscher Aktienindex) equity index for the period of 5th January 1988 to 29th December 2017. We compare the QAR plus Beta-*t*-EGARCH model with the classical AR (Box and Jenkins, 1976) plus *t*-GARCH (Engle, 1982; Bollerslev, 1986, 1987; Taylor, 1986) model. In Section 5, we present the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model (Harvey, 2013), which is used for daily AFN/USD (Afghan Afghani/ United States Dollar) currency exchange rate data for the period of 1st March 2007 to 7th July 2017. We compare the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model with the classical local level, classical dynamic seasonality plus GARCH model. In Section 6, we present the QVAR (quasi-vector autoregressive) model, which is a score-driven multivariate dynamic model of location (Harvey, 2013; Blazsek, Escribano and Licht, 2017). We use this model for monthly US inflation rate and US unemployment rate data for the period of 1st January 1948 to 1st December 2017. We compare the QVAR model with the classical VAR model (Sims, 1980, 1986; Sims, Goldfeld and Sachs, 1982; Bernanke, 1986; Lütkepohl, 2005). For all applications, we present the estimation results and the model diagnostics of the DCS models and their classical alternatives. Those results suggest that the statistical performance of each DCS model is superior to that of the classical model. We present conclusions in Section 7.

2. Literature review

In many practical applications that involve the estimation of time series models of financial or economic variables, model parameters are time-varying in order to provide an effective description of the data series. Cox (1981) classifies the time series models with time-varying parameters into two groups: observation-driven models and parameter-driven models.

For observation-driven models, the time variation of parameters is achieved by formulating those parameters as functions of observable dependent and exogenous variables. Examples of some classical observation-driven models from the body of literature are the ARMA (AR moving average) model (Box and Jenkins, 1976), the VAR model (Sims, 1980, 1986; Sims, Goldfeld and Sachs, 1982; Bernanke, 1986), the ARCH model (Engle, 1982), the GARCH model (Bollersev, 1986, 1987; Taylor, 1986), the EGARCH model (Nelson, 1991), the ACD (autoregressive conditional duration) model (Engle and Russell, 1998) and the ACI (autoregressive conditional intensity) model (Russell, 2001). An advantage of the observation-driven models is that their statistical inferences are relatively straightforward. All models in the class of DCS models are observation-driven time series models. DCS models can be related to classical observation-driven time series models, since classical observation-driven time series models are special cases of DCS models (e.g. ARMA, GARCH and VAR are special cases of QAR, Beta-t-GARCH and QVAR, respectively; Harvey, 2013).

For parameter-driven models, the parameters are stochastic processes with an internal error source. In those models, an unobserved error term updates the dynamic equation of each time-varying parameter. Some examples of the parameter-driven models from the body of literature are: the unobserved components (UC) model (Harvey, 1989), the stochastic volatility (SV) model (Harvey, Ruiz and Shephard, 1994; Harvey and Shephard, 1996), the stochastic conditional intensity (SCI) model (Bauwens and Hautsch, 2006; Koopman, Lucas and Monteiro, 2008) and the latent-factor count panel data model (Blazsek and Escribano, 2010, 2016). With respect to the parameter-driven models, we also refer to the book of Durbin and Koopman (2012). The estimation of parameter-driven models, in general, is more demanding than that of observation-driven models. Although the estimation of the UC models of Harvey (1989) can be performed by using the straightforward Kalman filter technique (Kalman, 1960) (that is also used for the estimation of the SV models in the works of Harvey, Ruiz and Shephard, 1994, and Harvey and Shephard, 1996), several more recent parameter-driven models are estimated by using simulation-based methods. Those simulation-based methods include, for example, the Markov chain Monte Carlo (MCMC) method (see an application in Chib, Nardari and Shephard, 2002) and the efficient importance sampling (EIS) technique (Richard and Zhang, 2007; see applications in Bauwens and Hautsch, 2006 and Blazsek and Escribano, 2010, 2016). Disadvantages of the simulation-based methods are that they may be time consuming, they may not be precise, and their asymptotic properties may be difficult to obtain. At least partly, these disadvantages motivated the works of Creal, Koopman and Lucas (2013) and Harvey (2013), for the introduction of the DCS models. DCS models can be related to parameter-driven models as follows. We obtain a DCS model from a parameter-driven model, by replacing the error term that updates each dynamic equation in the parameter-driven model by the score function (Harvey, 2013).

An important advantage of DCS models with respect to the classical time series models, is that DCS models are robust to extreme observations in the noise. This is due to the fact that the score function in a DCS model discounts extreme observations. Classical time series models (e.g. ARMA and GARCH) are not robust to extreme observations. For example, ARMA and GARCH use linear and quadratic transformations of the noise term, respectively. Thus, neither ARMA nor GARCH discount extreme observations (in fact, GARCH accentuates extreme observations due to the quadratic transformation). This property of GARCH is one of the motivations for the work of Harvey and Chakravarty (2008) that introduces the Beta-t-EGARCH model (to the best of our knowledge, Beta-t-EGARCH is the first DCS model in the body of literature).

3. Differences between classical and DCS time series models

In the first part of this section, we present the Gaussian signal plus noise model (Harvey, 1989) that is a classical parameter-driven time series model. The discussion of the Gaussian signal plus noise model is motivated by the fact that a specific DCS model, named as the ‘dynamic Student’s *t* location model’ (also named as the ‘QAR model’) (Harvey, 2013), can be directly related to it: We obtain the dynamic Student’s *t* location model if we replace the error term of the location equation in the Gaussian signal plus noise model by the score function. Thus, in the second part of this section, we present the QAR model. The presentation of the Gaussian signal plus noise and QAR models helps to highlight the differences in model formulations and in statistical inferences.

3.1. Gaussian signal plus noise model

The Gaussian signal plus noise model (Harvey, 1989) is a simple time series model that decomposes the dependent variable y_t into the local level component $c + \mu_t$ and the noise component v_t, according to the idea of signal extraction (Harvey, 1989). The Gaussian signal plus noise model is given by:

$$y_t = c + \mu_t + v_t \tag{1}$$

$$\mu_t = \phi\mu_{t-1} + \eta_t \quad (2)$$

for $t = 1, \ldots, T$, where each error term $v_t \sim NID(0, \sigma_v^2)$ and $\eta_t \sim NID(0, \sigma_\eta^2)$ is independent and identically distributed (i.i.d.), c is the constant parameter and ϕ is the first-order dynamic parameter. The condition $|\phi| < 1$ ensures that y_t is covariance stationary. With respect to the choice of the start value of μ_t, there are several possibilities: For example, one can use the unconditional mean $\mu_1 = E(\mu_t) = 0$ or, alternatively, one can estimate the initial condition μ_1 as a parameter. Substituting Equation (2) into Equation (1), we obtain the following reduced-form ARMA(1,1) model:

$$y_t = c(1 - \phi) + \phi y_{t-1} - \phi v_{t-1} + (v_t + \eta_t) \quad (3)$$

The Gaussian signal plus noise model is estimated by using the maximum likelihood (ML) method (Davidson and MacKinnon, 2003), for which the likelihood function is computed by using the Kalman filter technique (Kalman, 1960; Harvey, 1989). It is noteworthy that, when the data generating process (DGP) involves a heavy-tailed distribution for v_t, the Gaussian signal plus noise model will include extreme observations into the local level component $c + \mu_t$, instead of the irregular component v_t. Thus, the Gaussian signal plus noise model does not handle extreme observations in an appropriate way.

3.2. Dynamic Student's t location model

The 'dynamic Student's *t* location model' or 'QAR model' (Harvey, 2013) is in the class of DCS models. It is similar to the Gaussian signal plus noise model, since it also decomposes the dependent variable into the level and noise components, according to the signal extraction principle. The dynamic equation in the QAR(1) model is updated by the score function with respect to location, as follows:

$$y_t = c + \mu_t + v_t = \mu_t + \sigma\varepsilon_t \quad (4)$$

$$\mu_t = \phi\mu_{t-1} + \theta u_{\mu,t-1} \quad (5)$$

$$u_{\mu,t} = \frac{\nu\sigma\varepsilon_t}{\nu + \varepsilon_t^2} \quad (6)$$

where $\varepsilon_t \sim t(\nu)$ is an i.i.d. error term having the Student's *t* distribution (ν denotes the degrees of freedom parameter). As a consequence, the conditional distribution of y_t is the non-standardized Student's *t* distribution $y_t \sim t[c + \mu_t, \sigma, \nu]$, where $c + \mu_t$ is the dynamic location parameter, σ is the scale parameter and ν is the degrees of freedom parameter. The updating variable $u_{\mu,t}$ is proportional to the conditional score with respect to μ_t:

$$\frac{\partial \ln f(y_t | y_1, \ldots, y_{t-1})}{\partial \mu_t} = u_{\mu,t} \times \frac{\nu + 1}{\nu\sigma^2} \quad (7)$$

In Equations (4) and (5), c is the constant parameter and ϕ is the first-order dynamic parameter, respectively. The condition $|\phi| < 1$ ensures that y_t is covariance stationary. With respect to the choice of the start value of μ_t, there are several possibilities: For example, one can use the unconditional mean $\mu_1 = E(\mu_t) = 0$; alternatively, one can estimate the initial condition μ_1 as a parameter. The QAR(1) model is estimated by using the ML method (Davidson and MacKinnon, 2003; Harvey, 2013).

4. Dynamic models of expected return and volatility

In the first application, we compare the statistical performances of the AR(1) plus *t*-GARCH(1,1) and the QAR(1) plus Beta-*t*-EGARCH(1,1) models. We use daily log-return $y_t = \ln(p_t / p_{t-1})$ data from the DAX equity index, where p_t denotes the daily adjusted closing value of DAX. We control for possible serial correlation in the mean by using dynamic specifications for the expected return. We also control for possible serial correlation in the variance by using dynamic models of volatility. The estimation of DAX volatility is interesting for practitioners, for example, due to the following reasons: (i) When a financial or economic crisis impacts the markets, the volatility of DAX shares increases significantly. During those periods, the appropriate measurement and forecasting of volatility is important for financial investors and analysts. (ii) The value of financial derivatives on the DAX index is influenced by the volatility of the underlying DAX index. Examples of those financial derivatives are DAX futures and options, and exchange traded funds (ETFs) related to the DAX index.

4.1. Econometric models

Firstly, for the *t*-GARCH model, the daily return is modelled as

$$y_t = \mu_t + v_t = \mu_t + \lambda_t^{1/2}\varepsilon_t \tag{8}$$

for $t = 1, \ldots, T$, where the error term is $\varepsilon_t \sim t(\nu)$ i.i.d., having the Student's t distribution, μ_t is the time-varying conditional location parameter (i.e. conditional expected return), v_t denotes the unexpected return, and $\lambda_t^{1/2}$ is the time-varying conditional scale parameter driving conditional volatility. The conditional volatility of y_t is:

$$\sigma_t = \left(\lambda_t \times \frac{\nu}{\nu-2}\right)^{1/2} \tag{9}$$

The conditional location is specified according to the following AR(1) model:

$$\mu_t = c + \phi y_{t-1} \tag{10}$$

The conditional scale is specified according to the *t*-GARCH(1,1) model with leverage effects (Glosten, Jagannathan and Runkle, 1993), as follows:

$$\lambda_t = \omega + \beta\lambda_{t-1} + \alpha v_{t-1}^2 + \alpha^* v_{t-1}^2 I(v_{t-1} < 0) \tag{11}$$

where α^* is a measure of the leverage effects, and $I(\cdot)$ is the indicator function that takes the value one if the argument is true and zero otherwise. μ_t is initialized by using pre-sample data for y_t, and λ_t is initialized by using the parameter λ_0. For the AR(1) model, the conditional location is updated by a linear transformation of the new information represented by ε_t. For the *t*-GARCH model, the conditional scale is updated by a quadratic transformation of the new information that is represented by ε_t. Thus, the new information that arrives to the market is not discounted in these models (in fact, as aforementioned, the updating term of the *t*-GARCH model accentuates the impact of the new information).

Secondly, for the QAR plus Beta-*t*-EGARCH model, the daily return is modelled as:

$$y_t = \mu_t + v_t = \mu_t + \exp(\lambda_t)\varepsilon_t \tag{12}$$

where $\varepsilon_t \sim t(\nu)$ denotes the i.i.d. error term. The interpretation of μ_t and v_t is the same as for the AR plus GARCH model. The conditional volatility of y_t is given by:

$$\sigma_t = \exp(\lambda_t)\left(\frac{\nu}{\nu-2}\right)^{1/2} \tag{13}$$

The conditional location is specified according to the following QAR(1) model:

$$\mu_t = c + \phi\mu_{t-1} + \theta u_{\mu,t-1} \tag{14}$$

where $u_{\mu,t}$ is proportional to the conditional score with respect to μ_t that is given by

$$u_{\mu,t} = \frac{\nu\exp(\lambda_t)\varepsilon_t}{\nu+\varepsilon_t^2} \tag{15}$$

The conditional scale is specified according to the Beta-*t*-EGARCH(1,1) model with leverage effects:

$$\lambda_t = \omega + \beta\lambda_{t-1} + \alpha u_{\lambda,t-1} + \alpha^*\mathrm{sgn}(\mu_{t-1} - y_{t-1})(u_{\lambda,t-1} + 1) \tag{16}$$

where $\mathrm{sgn}(\cdot)$ is the signum function. Moreover, $u_{\lambda,t}$ is the conditional score with respect to λ_t that is given by

$$u_{\lambda,t} = \frac{(\nu+1)\varepsilon_t^2}{\nu+\varepsilon_t^2} - 1 \tag{17}$$

μ_t is initialized by using pre-sample data for y_t, and λ_t is initialized by using the parameter λ_0. An advantage of the use of both QAR(1) and Beta-*t*-EGARCH(1,1) models is that the updating terms $u_{\mu,t}$ and $u_{\lambda,t}$ discount the impact of the new information ε_t on location and scale, respectively.

4.2. Statistical inference

We estimate both models by using the ML method. The ML estimate of parameters is given by:

$$\widehat{\Theta}_{\mathrm{ML}} = \max_{\Theta} LL(y_1, \dots, y_T; \Theta) = \max_{\Theta} \textstyle\sum_{t=1}^{T} \ln f(y_t|y_1, \dots, y_{t-1}; \Theta) \tag{18}$$

where LL is the log-likelihood and $\ln f$ denotes the log conditional density function of the dependent variable. We obtain the ML estimates by numerical maximization at interior points of the parameter space. We use the gradient tolerance criterion of 10^{-5} for the numerical maximization. For several parameters, their transformed values are estimated. We compute the standard errors of those parameters by using the delta method (Davidson and MacKinnon, 2003).

The asymptotic properties of ML are ensured by using the following conditions: Firstly, for the AR(1) and the *t*-GARCH(1,1) with leverage effects equations, the covariance stationarity is supported if

$$c_{\mu1} = |\phi| < 1 \tag{19}$$

and

$$C_{\lambda1} = \alpha + \beta + \frac{\alpha^*}{2} < 1, \tag{20}$$

respectively. For the *t*-GARCH(1,1) with leverage effects equation, we also verify the condition of consistency and asymptotic normality of ML that is suggested in the work of Jensen and Rahbek (2004):

$$C_{\lambda2} = E\left[\frac{\beta}{\left(\alpha+\frac{\alpha^*}{2}\right)\varepsilon_t^2+\beta}\right] < 1 \tag{21}$$

If the expression within the square parentheses of the previous equation forms a covariance stationary time series, then the expected value can be estimated by using the sample average. We test the covariance stationarity of the expression within the square parentheses, by using the augmented Dickey-Fuller (1979) (ADF) unit root test. We find that all elements are covariance stationary (the ADF results are available from the authors upon request). Secondly, for the QAR(1) and Beta-*t*-EGARCH(1,1) equations, the covariance stationarity holds if

$$c_{\mu 1} = |\phi| < 1 \tag{22}$$

and

$$C_{\lambda 1} = |\beta| < 1, \tag{23}$$

respectively. For the QAR(1) and Beta-*t*-EGARCH(1,1) equations, we also verify two necessary conditions of consistency and asymptotic normality of ML that are suggested in the work of Harvey (2013):

$$c_{\mu 2} = \phi^2 - 2\phi\theta \frac{\nu}{\nu+3} + \theta^2 \frac{\nu(\nu^3+10\nu^2+35\nu+38)}{(\nu+1)(\nu+3)(\nu+5)(\nu+7)} < 1 \tag{24}$$

and

$$C_{\lambda 2} = \beta^2 - \alpha\beta \frac{4\nu}{\nu+3} + [\alpha^2 + (\alpha^*)^2] \times \frac{12\nu(\nu+1)(\nu+2)}{(\nu+7)(\nu+5)(\nu+3)} < 1, \tag{25}$$

respectively.

4.3. Data

We use data from the DAX Equity Index for the period of 5th January 1988 to 29th December 2017. The DAX Index is a market capitalization weighted average of the prices of 30 large German companies that are traded on the Frankfurt Stock Exchange (source: Yahoo Finance, http://www.finance.yahoo.com). We use the log-return time series $y_t = \ln(p_t/p_{t-1})$ with $t = 1, \ldots, T$, where T denotes the number of observations, and p_t denotes the adjusted closing value of the DAX Index for day t (for $t = 1$, we use pre-sample data of p_0). We present the descriptive statistics of y_t in Table 1. We present the evolution of DAX log-returns in Figure 1(a).

[APPROXIMATE LOCATION OF TABLE 1 AND FIGURE 1]

4.4. Estimation results

We present the parameter estimates, the ML conditions and the statistical performances of AR plus *t*-GARCH and QAR plus Beta-*t*-EGARCH in Table 1. We find that, for both models, the necessary conditions of the asymptotic properties of the ML estimator are supported (Table 1). For both models, we present the evolution of the conditional volatility σ_t in Figures 1(b) and 1(c), respectively.

For both classical and DCS models, in Figures 2(a) and 2(b) we present the treatment of extreme observations for location and scale, respectively, which is undertaken by the updating terms in the dynamic equations. In Figure 2(a), we present the updating terms of location as a function of the noise ε_t for AR(1) and QAR(1). For the AR(1) model, the new information is transformed according to a linear function. Hence, new information is not discounted by the AR model. On the other hand, for the QAR(1) model, the new information is discounted according to the non-linear score function. In Figure 2(b), we present the updating terms of log-scale as a function of the noise ε_t for *t*-GARCH(1,1) and Beta-*t*-EGARCH(1,1). For the *t*-GARCH(1,1) model, the new information is transformed according to a quadratic function. Hence, new information is not discounted by the *t*-GARCH(1,1) model (in fact, the new information is accentuated by the GARCH model). On the other hand, for the Beta-*t*-EGARCH(1,1) model, the new information is discounted according to the non-linear score function.

The statistical performance of both models is evaluated by using the following likelihood-based performance criteria: (i) LL, (ii) Akaike Information Criterion (AIC), (iii) Bayesian Information Criterion (BIC) and (iv) Hannan-Quinn Criterion (HQC) (Davidson and MacKinnon, 2003). We present these metrics in Table 1. All likelihood-based metrics suggest that QAR-Beta-*t*-EGARCH is superior to AR-*t*-GARCH. We conclude that the QAR plus Beta-*t*-EGARCH model improves the AR plus GARCH model for the estimation of the expected return and volatility of the DAX equity index.

[APPROXIMATE LOCATION OF FIGURE 2]

5. Dynamic local level, dynamic seasonality and dynamic volatility models

In the second application, we compare the classical local level, classical dynamic seasonality plus normal-GARCH model and the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model. We use data for the AFN/USD exchange rate. At least partly, this application is motivated by the fact that there is a very limited amount of literature on modelling the seasonality of the AFN/USD exchange rate p_t. We refer to the related work of Fry (1974), who analyses this pattern and attributes its causes mainly to the seasonality of agricultural exports from Afghanistan to abroad (with maximum exports, approximately, from September to November) and to the related subsequent USD inflows to Afghanistan, approximately, during December and January.

5.1. Econometric models

Firstly, for the classical local level, classical dynamic seasonality plus normal-GARCH(1,1) model, the daily value of the AFN/USD exchange rate is formulated as

$$p_t = \mu_t + s_t + v_t = \mu_t + s_t + \lambda_t^{1/2} \varepsilon_t \tag{26}$$

where $\varepsilon_t \sim N(0,1)$ is the i.i.d. noise term component of the exchange rate. The dynamic components are specified as follows. The local level component μ_t is

$$\mu_t = \mu_{t-1} + \theta v_{t-1} \tag{27}$$

which is updated by the first lag of the irregular component v_{t-1}. The seasonality component s_t is

$$s_t = D_t' \gamma_t \tag{28}$$

where $D_t = \left(D_{\mathrm{Jan},t}, \dots, D_{\mathrm{Dec},t}\right)'$ is a 12×1 vector of monthly dummy variables, and the 12×1 vector of dynamic parameters γ_t is

$$\gamma_t = \gamma_{t-1} + \kappa_t v_{t-1} \tag{29}$$

where κ_t is a 12×1 vector of dynamic parameters, for which each element of κ_t is parameterized as $\kappa_{jt} = \kappa_j$ if $D_{jt} = 1$ and $\kappa_{jt} = -\kappa_j/(12-1)$ if $D_{jt} = 0$. In this formulation, κ_{jt} with $j = \mathrm{Jan}, \dots, \mathrm{Dec}$ are parameters to be estimated. The parameterization ensures that the seasonality component s_t is centred at zero. The scale component λ_t is specified according to the normal-GARCH(1,1) model:

$$\lambda_t = \omega + \beta\lambda_{t-1} + \alpha v_{t-1}^2 \tag{30}$$

Secondly, for the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model, the daily value of the AFN/USD exchange rate is formulated as

$$p_t = \mu_t + s_t + v_t = \mu_t + s_t + \exp(\lambda_t)\varepsilon_t \tag{31}$$

where $\varepsilon_t \sim t[\exp(\nu) + 2]$ is the i.i.d. error term having the Student's t distribution with $\exp(\nu) + 2$ degrees of freedom. The dynamic components are specified as follows. The local level component μ_t is

$$\mu_t = \mu_{t-1} + \theta u_{\mu,t-1} \tag{32}$$

where the updating term is proportional to the score function with respect to μ_t that is given by

$$u_{\mu,t} = \frac{\exp(\lambda_t)\varepsilon_t}{\varepsilon_t^2 + \exp(\nu) + 2} \tag{33}$$

The seasonality component s_t is

$$s_t = D_t'\gamma_t \tag{34}$$

where $D_t = \left(D_{\text{Jan},t}, \ldots, D_{\text{Dec},t}\right)'$ is a 12×1 vector of monthly dummy variables, and the 12×1 vector of dynamic parameters γ_t is

$$\gamma_t = \gamma_{t-1} + \kappa_t u_{\mu,t-1} \tag{35}$$

where κ_t is a 12×1 vector of dynamic parameters, for which each element of κ_t is parameterized as $\kappa_{jt} = \kappa_j$ if $D_{jt} = 1$ and $\kappa_{jt} = -\kappa_j/(12-1)$ if $D_{jt} = 0$. In this formulation, κ_{jt} with $j = \text{Jan}, \ldots, \text{Dec}$ are parameters to be estimated. The parameterization ensures that the seasonality component s_t is centred at zero. The scale component λ_t is specified according to the Beta-*t*-EGARCH model:

$$\lambda_t = \omega + \beta\lambda_{t-1} + \alpha u_{\lambda,t-1} \tag{36}$$

where the score function is given by:

$$u_{\lambda,t} = \frac{[\exp(\nu) + 3]\varepsilon_t^2}{\exp(\nu) + 2 + \varepsilon_t^2} - 1 \tag{37}$$

5.2. Statistical inference

We estimate both models by using the ML method. The ML estimate of parameters is given by:

$$\widehat{\Theta}_{\text{ML}} = \max_{\Theta} LL(p_1, \ldots, p_T; \Theta) = \max_{\Theta} \sum_{t=1}^{T} \ln f(p_t | p_1, \ldots, p_{t-1}; \Theta) \tag{38}$$

where LL is the log-likelihood and $\ln f$ denotes the log conditional density function of the dependent variable. We obtain the ML estimates by numerical maximization at interior points of the parameter space. We use the gradient tolerance criterion of 10^{-5} for the numerical maximization. For several parameters, the transformed values of parameters are estimated. We compute the standard errors of those parameters by using the delta method (Davidson and MacKinnon, 2003).

The asymptotic properties of ML are ensured by the following conditions: For the normal-GARCH(1,1) equation, covariance stationarity is supported if

$$C_{\lambda 1} = \alpha + \beta < 1 \tag{39}$$

In addition, for the normal-GARCH(1,1) equation, we also verify the condition of consistency and asymptotic normality of the ML estimates that is suggested in the work of Jensen and Rahbek (2004):

$$C_{\lambda 2} = E\left(\frac{\beta}{\alpha\varepsilon_t^2 + \beta}\right) < 1 \tag{40}$$

We estimate the expected value in the previous equation by using the sample average of the expression within the squared parentheses (we validate the use of the sample average estimator by the ADF unit root test). For the Beta-t-EGARCH(1,1) equation, covariance stationarity is supported if

$$C_{\lambda 1} = |\beta| < 1 \tag{41}$$

In addition, for the Beta-t-EGARCH(1,1) equation, we also verify the following condition of consistency and asymptotic normality of the ML, which is suggested in the work of Harvey (2013):

$$C_{\lambda 2} = \beta^2 - \alpha\beta \frac{4[\exp(\nu)+2]}{\exp(\nu)+5} + \alpha^2 \times \frac{12[\exp(\nu)+2][\exp(\nu)+3][\exp(\nu)+4]}{[\exp(\nu)+9][\exp(\nu)+7][\exp(\nu)+5]} < 1 \tag{42}$$

It is noteworthy that the local level component and the dynamic seasonality component are formulated according to a unit root process (i.e. a first-order dynamic models, for which the dynamic parameters are set to one). For those components, the asymptotic properties of the ML estimator hold, since the dynamic parameter is set to one, rather than estimated (Harvey, 2013).

5.3. Data

We use the information of the AFN/USD exchange rate for the period of 1st March 2007 to 7th July 2017 (source: Bloomberg). The models are estimated for the daily closing exchange rate p_t for days $t = 1, \ldots, T$, where T denotes the number of observations. We present the descriptive statistics of p_t in Table 2. We present the evolution of the AFN/USD exchange rate in Figure 3(a).

[APPROXIMATE LOCATION OF TABLE 2 AND FIGURE 3]

5.4. Estimation results

We present the parameter estimates, the ML conditions and the statistical performances of the classical local level, classical dynamic seasonality plus normal-GARCH model and the score-driven local level, score-driven seasonality plus Beta-t-EGARCH model in Table 2. For both models, we find that the conditions of the asymptotic properties of the ML estimator are supported (Table 2). For both models, we present the local level components of the classical and the DCS model in Figures 3(b) and 3(c), respectively. We also present the seasonality components of the classical and the DCS model in Figures 3(d) and 3(e), respectively.

For both classical and DCS models, in Figures 4(a) and 4(b), we present the treatment of extreme observations for location and scale, respectively, which is undertaken by the updating terms in the dynamic equations. In Figure 4(a), we present the updating terms of location as a function of the noise term ε_t. For the classical model, the new information is transformed according to a linear function. Hence, new information is not discounted by the classical model. On the other hand, for the DCS model, the new information is discounted according to the non-linear score function. In Figure 4(b), we present the updating terms of log-scale as a function of the noise ε_t for normal-GARCH(1,1) and Beta-*t*-EGARCH(1,1). For the normal-GARCH(1,1) model, the new information is transformed according to a quadratic function. Hence, new information is not discounted by the normal-GARCH(1,1) model. On the other hand, for the Beta-*t*-EGARCH(1,1) model, the new information is discounted according to the non-linear score function.

The statistical performance of both models is evaluated by using the following likelihood-based performance criteria: LL, AIC, BIC and HQC (Table 2). All likelihood-based metrics suggest that the classical local level, classical dynamic seasonality plus normal-GARCH model is superior to the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model. Thus, the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model improves the classical local level, classical dynamic seasonality plus normal-GARCH model for the estimation of local level, seasonality and volatility of the AFN/USD exchange rate.

[APPROXIMATE LOCATION OF FIGURE 4]

6. Multivariate dynamic models of location

In the third application, we present an application of the QVAR(1) multivariate DCS model (Harvey, 2013; Blazsek, Escribano and Licht, 2017). We use QVAR(1) in order to study the dynamic interaction effects between the US inflation and unemployment rates. We also compare the statistical performances of QVAR(1) and VAR(1) models. This analysis is motivated by the work of Nordhaus (1975), who proposes a macroeconomic model of inflation and unemployment rates related to government-policies, for which the objective of the government is to win the next elections. That paper presents that the objective of winning the next elections may create political cycles of economic recessions and economic expansions: After winning the elections, the government tends to establish policies that keep the inflation rate at a low level, at the expense of a high unemployment rate. Subsequently, during the period preceding the next elections, government policies are modified to increase the level of inflation and decrease the unemployment rate, hence, to increase the probability of winning the next elections. The governments pursue these economic policies because, on the date of the elections, the voters discount the costs of past economic outcomes (e.g. high unemployment rate) (see also Findley, 2015, for a related discussion and for an extension of the model of Nordhaus, 1975).

6.1. Econometric models

Firstly, the VAR(1) model for the vector of dependent variables $(y_{1t}, \dots, y_{Kt})'$ is given by:

$$y_t = \mu_t + v_t = \mu_t + \Omega^{-1}\varepsilon_t \quad (43)$$

μ_t is the conditional mean of $y_t|(y_1, \dots, y_{t-1})$ that is specified as

$$\mu_t = c + \Phi y_{t-1} \quad (44)$$

where c is a $K \times 1$ vector of constant parameters and Φ is a $K \times K$ matrix of dynamic parameters. v_t is a $K \times 1$ vector of contemporaneously correlated reduced-form error terms, ε_t is a $K \times 1$ vector of contemporaneously uncorrelated structural-form error terms, and Ω^{-1} is a $K \times K$ lower-triangular scaling matrix. The process is initialized by using $\mu_1 = E(y_t) = (I_K - \Phi)^{-1}c$. For this model, $E(y_t|y_1, \dots, y_{t-1}) = \mu_t$. The structural-form linear vector moving average (VMA) representation of y_t is

$$y_t = \sum_{j=0}^{\infty} \Phi^j c + \sum_{j=0}^{\infty} \Phi^j \Omega^{-1} \varepsilon_{t-j} \quad (45)$$

From this representation, we obtain that the impulse response function $IRF_j = \frac{\partial y_{t+j}}{\partial \varepsilon_t}$ for $j = 0, 1, \dots, \infty$ is

$$IRF_j = \Phi^j \Omega^{-1} \tag{46}$$

Secondly, the QVAR(1) model for the vector of dependent variables $(y_{1t}, \dots, y_{Kt})'$ is given by:

$$y_t = c + \mu_t + v_t = c + \mu_t + \Omega^{-1}\varepsilon_t \tag{47}$$

where $v_t \sim t_K(0, \Sigma, \nu)$ has a multivariate Student's t distribution that is the multivariate i.i.d. reduced-form error term. Σ is positive definite and $\nu > 2$. The interpretation of all parameters and variables of QVAR(1) is the same as that of VAR(1). μ_t is specified as

$$\mu_t = \Phi\mu_{t-1} + \Psi u_{\mu,t-1} \tag{48}$$

where $u_{\mu,t}$ is proportional to the conditional score with respect to μ_t. The process is initialized by setting $\mu_1 = E(\mu_t) = 0_{K\times 1}$. For this model, $E(y_t|y_1, \dots, y_{t-1}) = c + \mu_t$. The log conditional density of y_t is

$$\ln f(y_t|y_1, \dots, y_{t-1}) = \ln\Gamma\left(\frac{\nu+K}{2}\right) - \ln\Gamma\left(\frac{\nu}{2}\right) - \frac{K}{2} - \frac{1}{2}\ln|\Sigma| - \frac{\nu+K}{2}\ln\left(1 + \frac{v_t'\Sigma^{-1}v_t}{\nu}\right) \tag{49}$$

The partial derivative of the log of the conditional density with respect to μ_t is

$$\frac{\partial \ln f(y_t|y_1,\dots,y_{t-1})}{\partial \mu_t} = \frac{\nu+K}{\nu}\Sigma^{-1} \times \left(1 + \frac{v_t'\Sigma^{-1}v_t}{\nu}\right) v_t = \frac{\nu+K}{\nu}\Sigma^{-1} \times u_{\mu,t} \tag{50}$$

The second equality of the previous equation defines the score function $u_{\mu,t}$ that updates the conditional mean of y_t. The structural-form nonlinear VMA representation of y_t is

$$y_t = c + \left\{\sum_{j=0}^{\infty} \Phi^j \Psi[(\nu-2)\nu]^{1/2}\Omega^{-1}\frac{\varepsilon_{t-1-j}}{\nu-2+\varepsilon_{t-1-j}'\varepsilon_{t-1-j}}\right\} + \left(\frac{\nu}{\nu-2}\right)^{1/2}\Omega^{-1}\varepsilon_t \tag{51}$$

From this representation, we obtain that the impulse response function $IRF_j = \frac{\partial y_{t+j}}{\partial \varepsilon_t}$ is given by:

$$IRF_{jt} = \left(\frac{\nu}{\nu-2}\right)^{1/2}\Omega^{-1} \text{ for } j = 0 \tag{52}$$

$$IRF_{jt} = \Phi^j \Psi[(\nu-2)\nu]^{1/2}\Omega^{-1}D_{t-1-j} \text{ for } j = 1, \dots, \infty \tag{53}$$

$$D_t = \frac{\partial \frac{\varepsilon_t}{\nu-2+\varepsilon_t'\varepsilon_t}}{\partial \varepsilon_t} \tag{54}$$

As can be seen in the previous equations, IRF_{jt} is time-dependent for $j = 1, \dots, \infty$. In this paper, we use its unconditional mean to measure the dynamic interaction effects between inflation and unemployment:

$$IRF_j = E(IRF_{jt}) = \Phi^j \Psi[(\nu-2)\nu]^{1/2}\Omega^{-1}E(D_{t-1-j}) \tag{55}$$

If all elements of D_{t-1-j} form covariance stationary time series, then $E(D_{t-1-j})$ can be estimated by using the sample average (we validate the use of the sample average estimator by the ADF unit root test).

6.2. Statistical inference

We estimate VAR(1) and QVAR(1) by using the ML method (Davidson and MacKinnon, 2003), respectively. For both models, the ML estimates are:

$$\widehat{\Theta}_{\mathrm{ML}} = \max_{\Theta} LL(y_1, \ldots, y_T; \Theta) = \max_{\Theta} \sum_{t=1}^{T} \ln f(y_t | y_1, \ldots, y_{t-1}; \Theta) \tag{56}$$

where $\ln f$ denotes the log conditional density function of y_t. We obtain the ML estimates by numerical maximization at interior points of the parameter space. We use the gradient tolerance criterion of 10^{-5} for the numerical maximization. For several parameters, the transformed values of parameters are estimated. We compute the standard errors of those parameters by using the delta method (Davidson and MacKinnon, 2003).

For both VAR(1) and QVAR(1), the covariance stationarity of y_t is ensured by the fact that the modulus of all eigenvalues of Φ is less than one (we denote the maximum modulus of all eigenvalues with C_μ). Related to this, we refer to the results of Lütkepohl (2005, Chapter 11) and Harvey (2013), respectively. For VAR(1), this condition ensures that the ML estimator is consistent and that it is asymptotically normal. For QVAR(1), we refer to additional conditions of consistency and asymptotic normality of the ML estimator that are demonstrated in the work of Blazsek, Escribano and Licht (2017).

6.3. Data

We use monthly data on the US inflation rate y_{1t} and the US unemployment rate y_{2t} for the period of 1st January 1948 to 1st December 2017 (source: Federal Reserve Bank of St. Louis, https://www.stlouisfed.org/). The models are estimated for the time series $y_t = (y_{1t}, y_{2t})'$ for $t = 1, \ldots, T$. We present some descriptive statistics of y_{1t} and y_{2t} in Table 3. We present the evolution of y_{1t} and y_{2t} in Figure 5.

[APPROXIMATE LOCATION OF TABLE 3 AND FIGURE 5]

6.4. Estimation results

We present the parameter estimates, the ML conditions and the statistical performances of the VAR(1) and QVAR(1) models in Table 4. For both models, we find that all conditions of covariance stationarity are supported (see Table 4). It is noteworthy that for QVAR(1) all conditions of consistency and asymptotic normality of the ML estimator (Blazsek, Escribano and Licht, 2017) are satisfied (we do not report the corresponding statistics in this paper, but those results are available from the authors upon request).

For VAR(1) and QVAR(1), we present the IRFs in Figures 6 and 7, respectively. For VAR(1), the IRF shows non-significant dynamic effects of unemployment shocks on inflation and negative significant dynamic effects of inflation shocks on unemployment (Figure 6). Thus, the IRF estimates of VAR(1) do not support completely the theory presented in Nordhaus (1975). On the other hand, for QVAR(1), the IRF shows significant negative dynamic effects of unemployment shocks on inflation and also negative dynamic effects of inflation shocks on unemployment (Figure 7). Thus, the IRF estimates of QVAR(1) support the theory of Nordhaus (1975), with respect to the negative interaction effects between the US inflation and unemployment rates.

The statistical performance of both models is evaluated by using the following likelihood-based performance criteria: LL, AIC, BIC and HQC. We present these metrics in Table 4. All likelihood-based metrics suggest that QVAR(1) is superior to VAR(1) (Table 4). We conclude that the QVAR(1) model improves the classical VAR(1) model for the estimation of interaction effects for US inflation and US unemployment rates.

[APPROXIMATE LOCATION OF TABLE 4 AND FIGURES 6-7]

7. Conclusions

We have provided a review of the DCS class of time series models, which have recently appeared in the body literature in economics. DCS models can be directly related to several classical time series models. To highlight the main differences between DCS and classical models, we have compared two simple models of location: the Gaussian signal plus noise model (i.e. a classical time series model) and the QAR model (i.e. a DCS model). Perhaps, the main difference between DCS and classical models is that DCS models are robust to outliers (i.e. extreme values in the irregular component). One of the consequences of this robustness property is that the ML conditions may be satisfied for DCS models, while the same conditions may not be satisfied for classical time series models, due to outliers. An interesting property of DCS models is that those models are generalizations of classical time series models (for example, QAR and QVAR are generalizations of ARMA and VAR, respectively). As a consequence, in many cases, DCS models provide a better fit to time series data than classical time series models. We have presented this superior statistical performance in three examples that involve data from the following variables: (i) DAX Index; (ii) AFN/USD exchange rate; (iii) US inflation rate and US unemployment rate. We have found that the DCS models are superior to the classical time series models.

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Table 1. Descriptive statistics, parameter estimates and model diagnostics

Descriptive statistics		t-GARCH parameters		Beta-t-EGARCH parameters	
Start date	1st January 1988	c_1	0.0006*** (0.0001)	c_1	0.0004 (0.0003)
End date	29th December 2017	φ_1	-0.0103 (0.0110)	φ_1	0.2829 (0.5625)
Sample size	7,742	θ_1	NA	θ_1	-0.0216 (0.0156)
Minimum	-0.1409	ω_1	0.0000*** (0.0000)	ω_1	-0.0768*** (0.0127)
Maximum	0.1080	α_1	0.0163*** (0.0041)	α_1	0.0413*** (0.0033)
Average	0.0003	α_1^*	0.0878*** (0.0110)	α_1^*	0.0276*** (0.0031)
Median	0.0003	β_1	0.9042*** (0.0089)	β_1	0.9837*** (0.0027)
SD	0.0138	λ_0	0.0005** (0.0002)	λ_0	-3.7916*** (0.2606)
Skewness	-0.2529	ν	7.3074*** (0.7572)	ν	7.0991*** (0.6665)
Excess kurtosis	6.2139				
Model diagnostics		$C_{\mu 1}$	0.0103	$C_{\mu 1}$	0.2829
				$C_{\mu 2}$	0.0889
		$C_{\lambda 1}$	0.9644	$C_{\lambda 1}$	0.9837
		$C_{\lambda 2}$	0.9302	$C_{\lambda 2}$	0.8623
		LL	3.0615	LL	**3.0646**
		AIC	-6.1209	AIC	**-6.1269**
		BIC	-6.1138	BIC	**-6.1189**
		HQC	-6.1185	HQC	**-6.1241**

Notes: Standard deviation (SD); not available (NA); log-likelihood (LL); Akaike information criterion (AIC); Bayesian information criterion (BIC), Hannan-Quinn criterion (HQC). Bold numbers indicate superior model performance. For the parameter estimates, standard errors are reported in parentheses. ** and *** indicate significance at the 5% and 1% levels, respectively.

Table 2. Descriptive statistics, parameter estimates and model diagnostics

Descriptive statistics		Local level-seasonal-GARCH parameters		Local level-seasonal-Beta-t-EGARCH parameters	
Start date	1st March 2007	κ	0.7166*** (0.0224)	κ	3.7757*** (0.1307)
End date	7th July 2017	ν	NA	ν	-0.6121*** (0.1482)
Sample size	2,622	κ_{Jan}	0.0815 (0.0768)	κ_{Jan}	0.2214*** (0.0442)
Minimum	44.3400	κ_{Feb}	0.0498** (0.0207)	κ_{Feb}	0.5541*** (0.0381)
Maximum	69.5400	κ_{Mar}	0.0677** (0.0344)	κ_{Mar}	0.2666*** (0.0333)
Average	54.4609	κ_{Apr}	-0.0326*** (0.0048)	κ_{Apr}	-0.1227*** (0.0153)
Median	51.6350	κ_{May}	0.0597*** (0.0108)	κ_{May}	0.0661** (0.0272)
SD	7.2344	κ_{Jun}	0.0985*** (0.0280)	κ_{Jun}	-0.2126*** (0.0359)
Skewness	0.7346	κ_{Jul}	-0.1310*** (0.0080)	κ_{Jul}	-0.4106*** (0.0384)
Excess kurtosis	-0.6795	κ_{Aug}	0.1347*** (0.0093)	κ_{Aug}	0.0287 (0.0362)
		κ_{Sep}	0.0616*** (0.0237)	κ_{Sep}	0.0635 (0.0726)
		κ_{Oct}	0.0301*** (0.0031)	κ_{Oct}	0.0213 (0.0160)
		κ_{Nov}	-0.1277*** (0.0183)	κ_{Nov}	-0.1369* (0.0754)
		κ_{Dec}	-0.0542*** (0.0092)	κ_{Dec}	-0.2053*** (0.0237)
		ω	0.0045*** (0.0002)	ω	-0.3988*** (0.0224)
		β	0.6691*** (0.0088)	β	0.8021*** (0.0800)
		α	0.3481*** (0.0128)	α	0.5000*** (0.0179)
		λ_0	0.0275 (0.0543)	λ_0	-1.5310 (0.9334)
Model diagnostics		C_μ	0.6691	C_μ	0.8021
		C_λ	0.8674	C_λ	0.3367
		LL	0.2487	LL	**0.5761**
		AIC	0.4844	AIC	**-1.1384**
		BIC	-0.4463	BIC	**-1.0981**
		HQC	-0.4706	HQC	**-1.1238**

Notes: Standard deviation (SD); not available (NA); log-likelihood (LL); Akaike information criterion (AIC); Bayesian information criterion (BIC), Hannan-Quinn criterion (HQC). Bold numbers indicate superior model performance. For the parameter estimates, standard errors are reported in parentheses. *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Table 3. Descriptive statistics

Descriptive statistics	Inflation rate	Unemployment rate
Start date	1st February 1948	1st February 1948
End date	1st December 2017	1st December 2017
Sample size	839	839
Minimum	-21.2466	2.5000
Maximum	21.7195	10.8000
Average	3.3706	5.7926
Median	2.9268	5.6000
SD	4.0137	1.6337
Skewness	0.4275	0.6180
Excess kurtosis	3.8362	0.1055

Notes: Standard deviation (SD).

Table 4. Parameter estimates and model diagnostics

VAR(1) parameters		QVAR(1) parameters	
c_1	0.0069 (0.1152)	c_1	-0.0111 (0.0280)
c_2	0.0003 (0.0067)	c_2	-0.0026 (0.0125)
$\Phi_{1,1}$	-0.3282*** (0.0313)	$\Phi_{1,1}$	0.1857*** (0.0494)
$\Phi_{1,2}$	0.6678 (0.5601)	$\Phi_{1,2}$	-1.1699 (0.7653)
$\Phi_{2,1}$	0.0013 (0.0020)	$\Phi_{2,1}$	0.0049 (0.0044)
$\Phi_{2,2}$	0.1218*** (0.0320)	$\Phi_{2,2}$	0.8626*** (0.0299)
$\Omega_{1,1}^{-1}$	3.3936*** (0.0889)	$\Omega_{1,1}^{-1}$	2.5017*** (0.0784)
$\Omega_{2,1}^{-1}$	-0.0170** (0.0068)	$\Omega_{2,1}^{-1}$	-0.0078 (0.0055)
$\Omega_{2,2}^{-1}$	0.2071*** (0.0054)	$\Omega_{2,2}^{-1}$	0.1612*** (0.0050)
		ν	6.2942*** (0.7629)
		$\kappa_{1,1}$	-0.8639*** (0.0630)
		$\kappa_{1,2}$	0.2753 (0.7531)
		$\kappa_{2,1}$	0.0011 (0.0039)
		$\kappa_{2,2}$	0.2388*** (0.0384)
Model diagnostics			
C_μ	0.3302	C_μ	0.8540
LL	-2.4852	LL	**-2.2472**
AIC	4.9918	AIC	**4.5279**
BIC	5.0427	BIC	**4.6069**
HQC	5.0113	HQC	**4.5582**

Notes: Log-likelihood (LL); Akaike information criterion (AIC); Bayesian information criterion (BIC); Hannan-Quinn criterion (HQC). Bold numbers indicate superior model performance. Standard errors are reported in parentheses. ** and *** indicate significance at the 5% and 1% levels, respectively.

Figure 1(a). DAX log-return

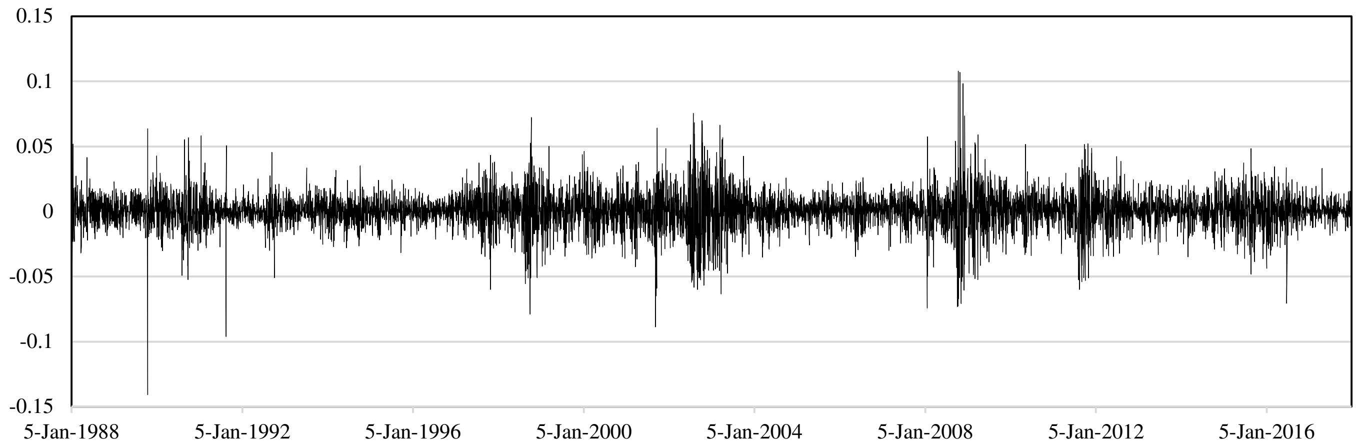


Figure 1(b). *t*-GARCH volatility

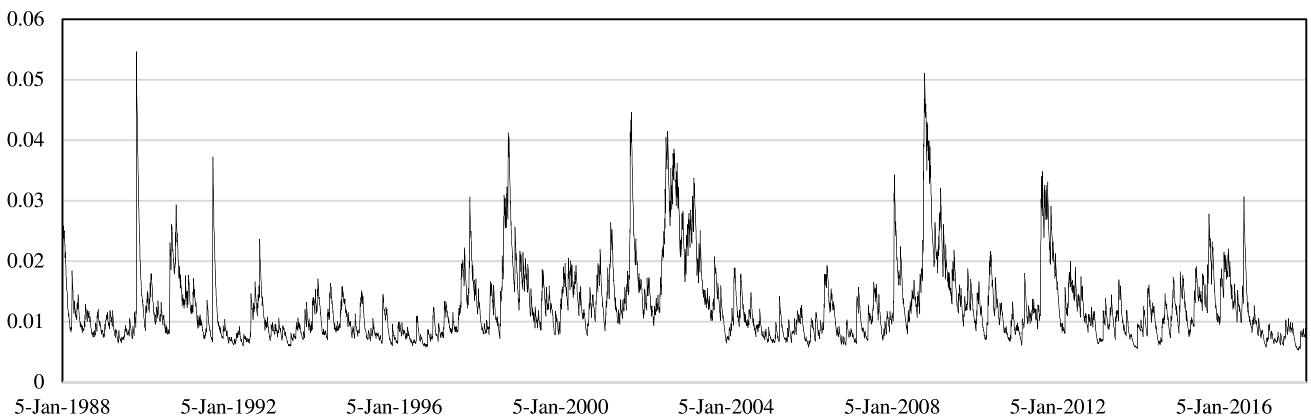


Figure 1(c). Beta-*t*-EGARCH volatility

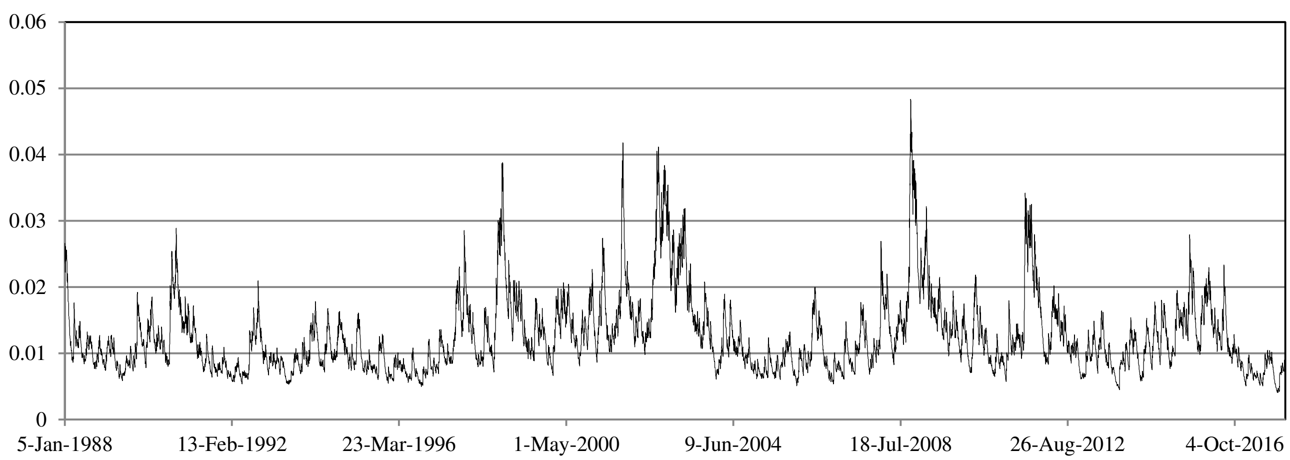


Figure 1. Evolution of DAX log-return, DAX volatility for *t*-GARCH and DAX volatility for Beta-*t*-EGARCH

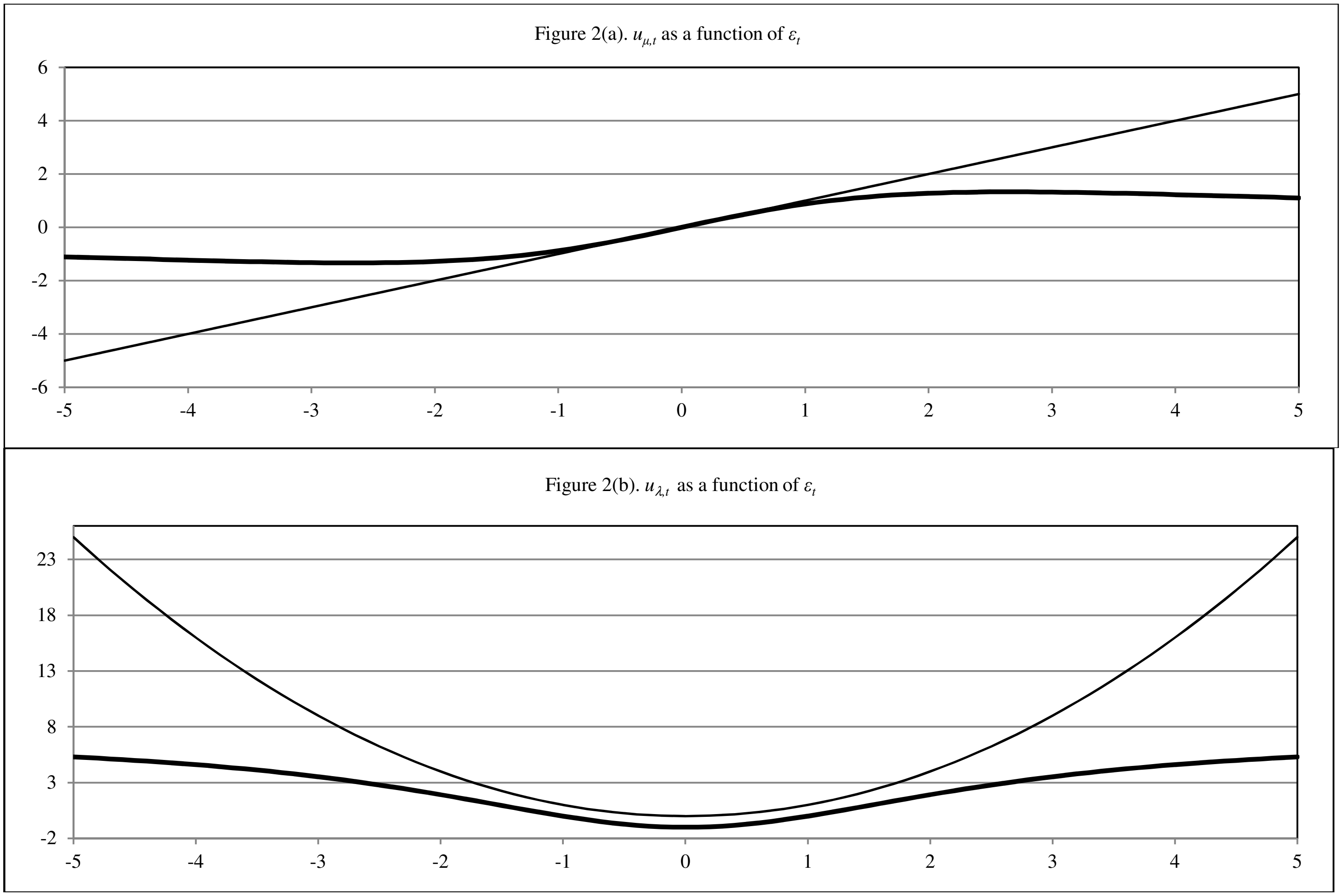


Figure 2. Updating terms of location and scale for AR plus GARCH and QAR plus Beta-t-EGARCH *Notes*: The updating terms for AR and *t*-GARCH are with solid thin lines in Figures 2(a) and 2(b), respectively. The updating terms for QAR and Beta-*t*-EGARCH are with solid thick lines in Figures 2(a) and 2(b), respectively.

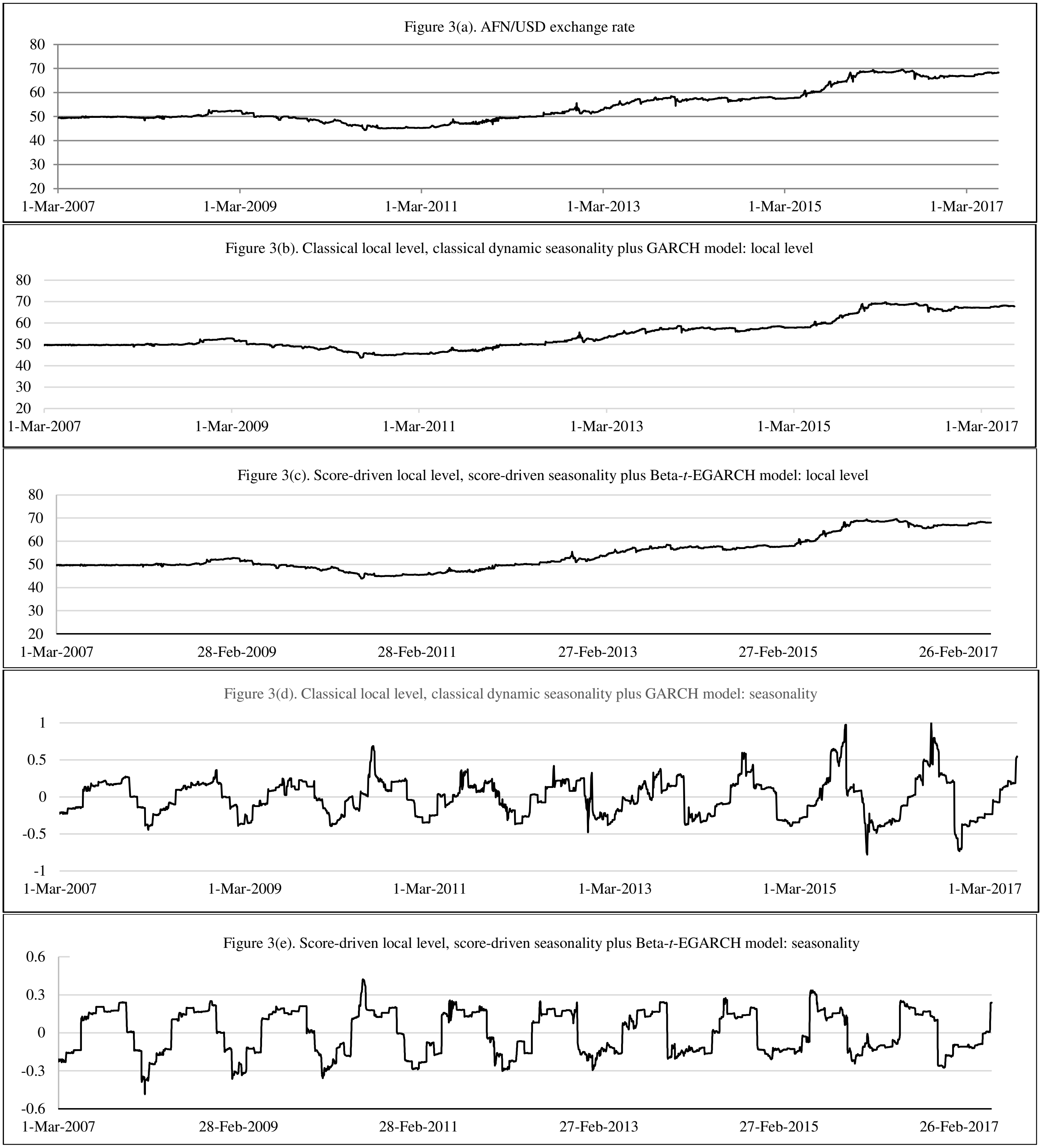
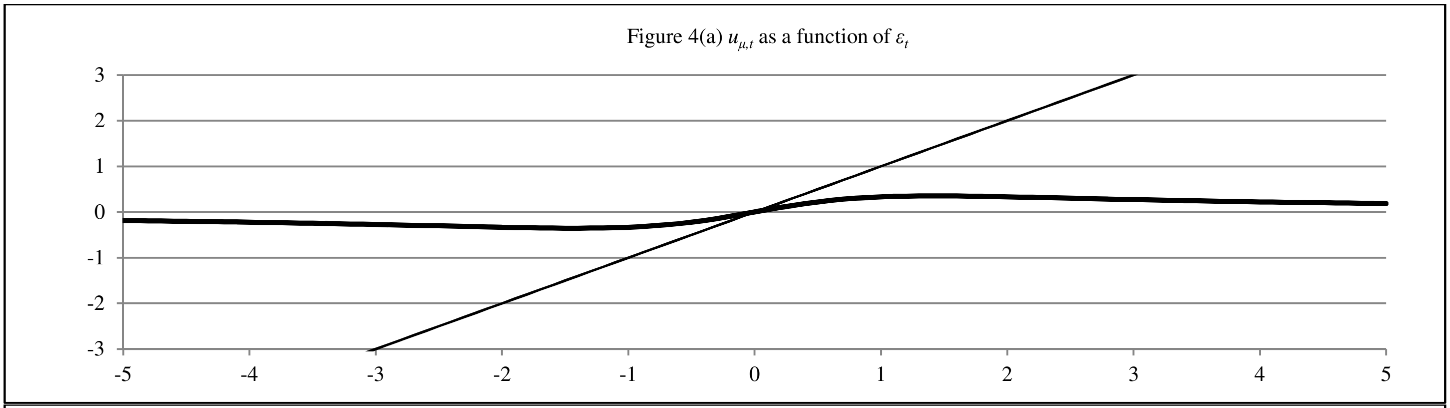


Figure 3. Evolution of AFN/USD, local level components and seasonality components



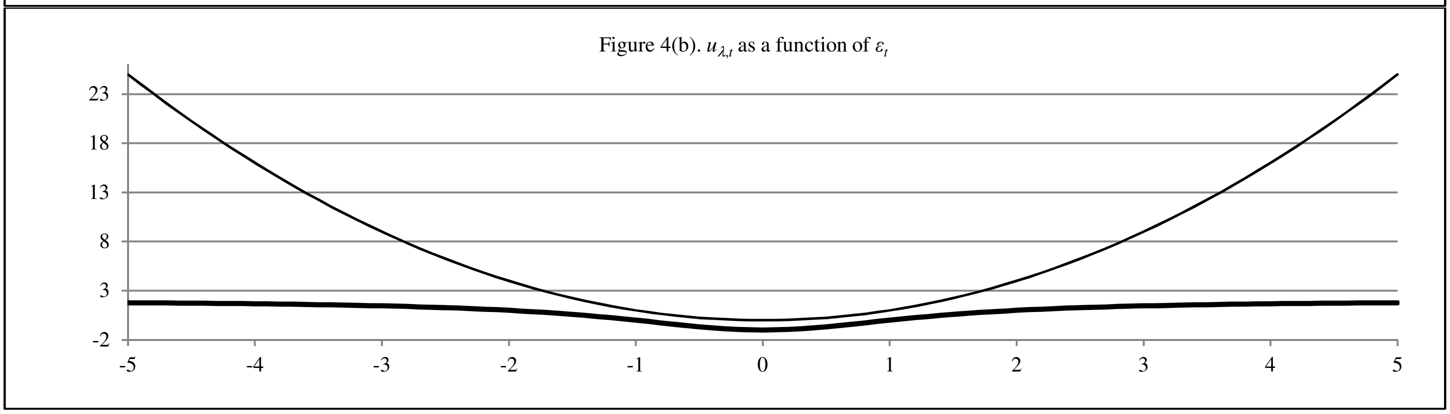


Figure 4. Updating terms of location and scale for the classical local level, classical dynamic seasonality plus GARCH model and the score-driven local level, score-driven seasonality plus Beta-*t*-EGARCH model *Notes*: The updating terms for the classical model are with solid thin lines. The updating terms for the DCS model are with solid thick lines.

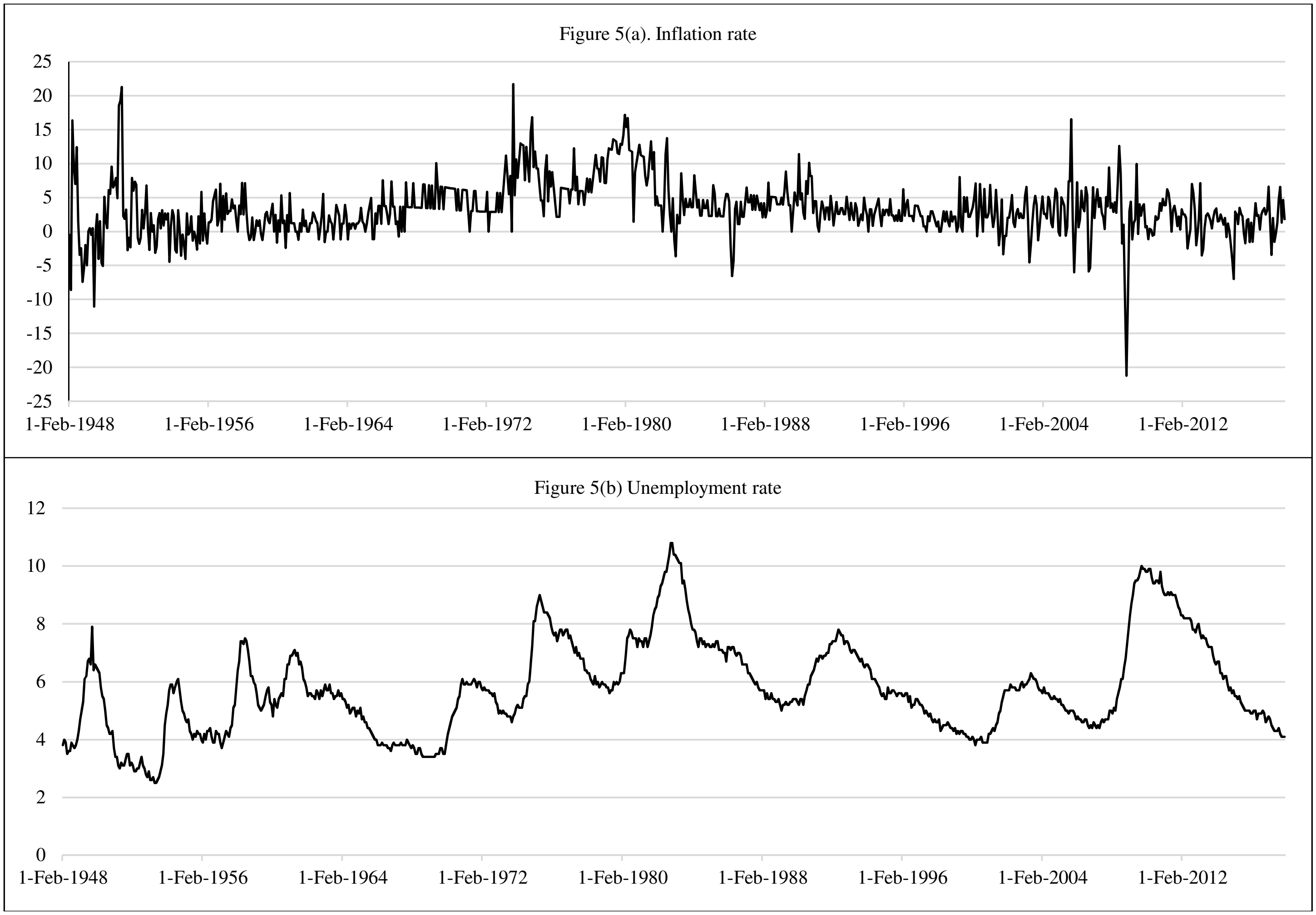


Figure 5. Evolution of the US inflation and unemployment rates

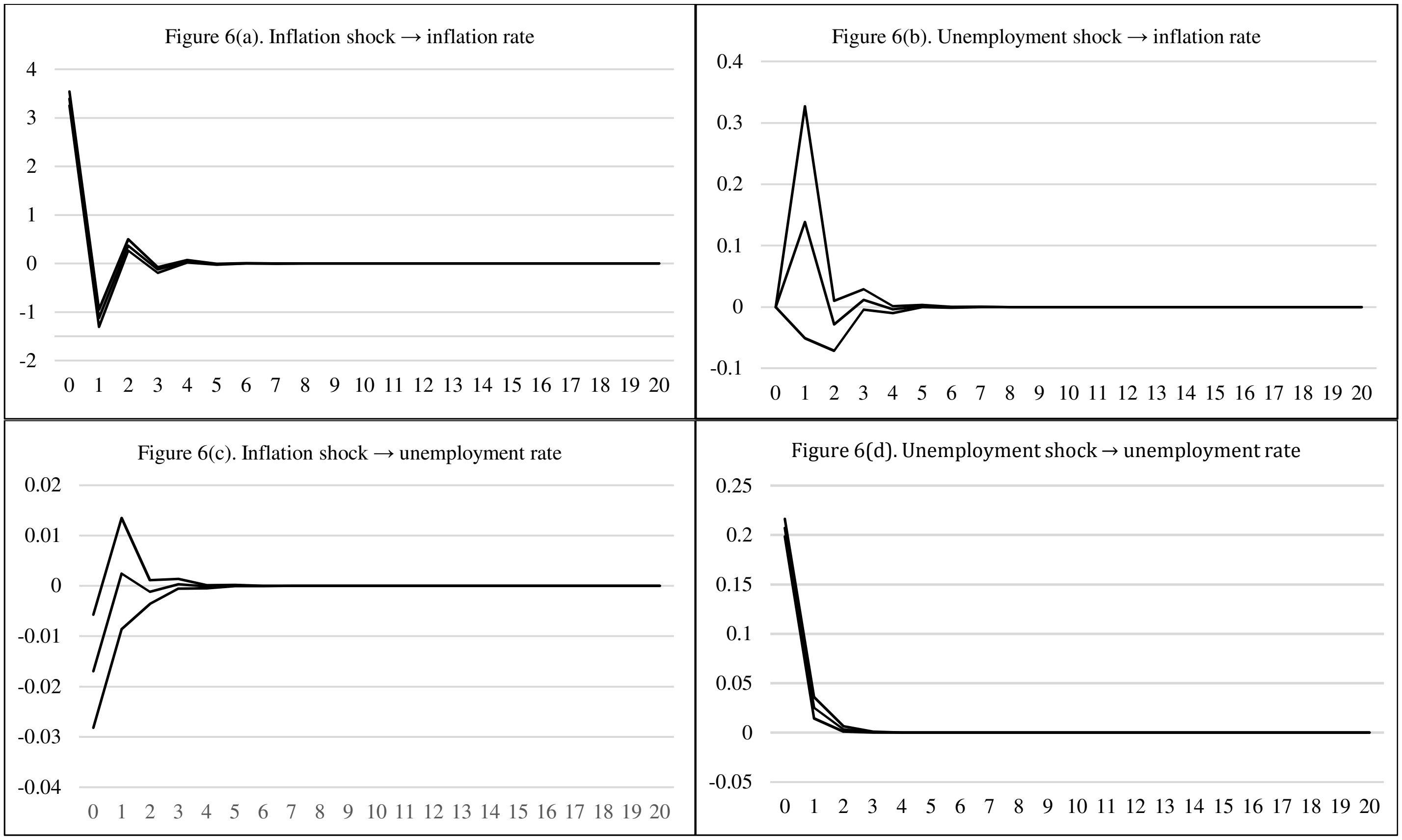


Figure 6. IRF for the VAR(1) model *Notes*: We present the IRF estimate up to 20 months with the 90% confidence interval.

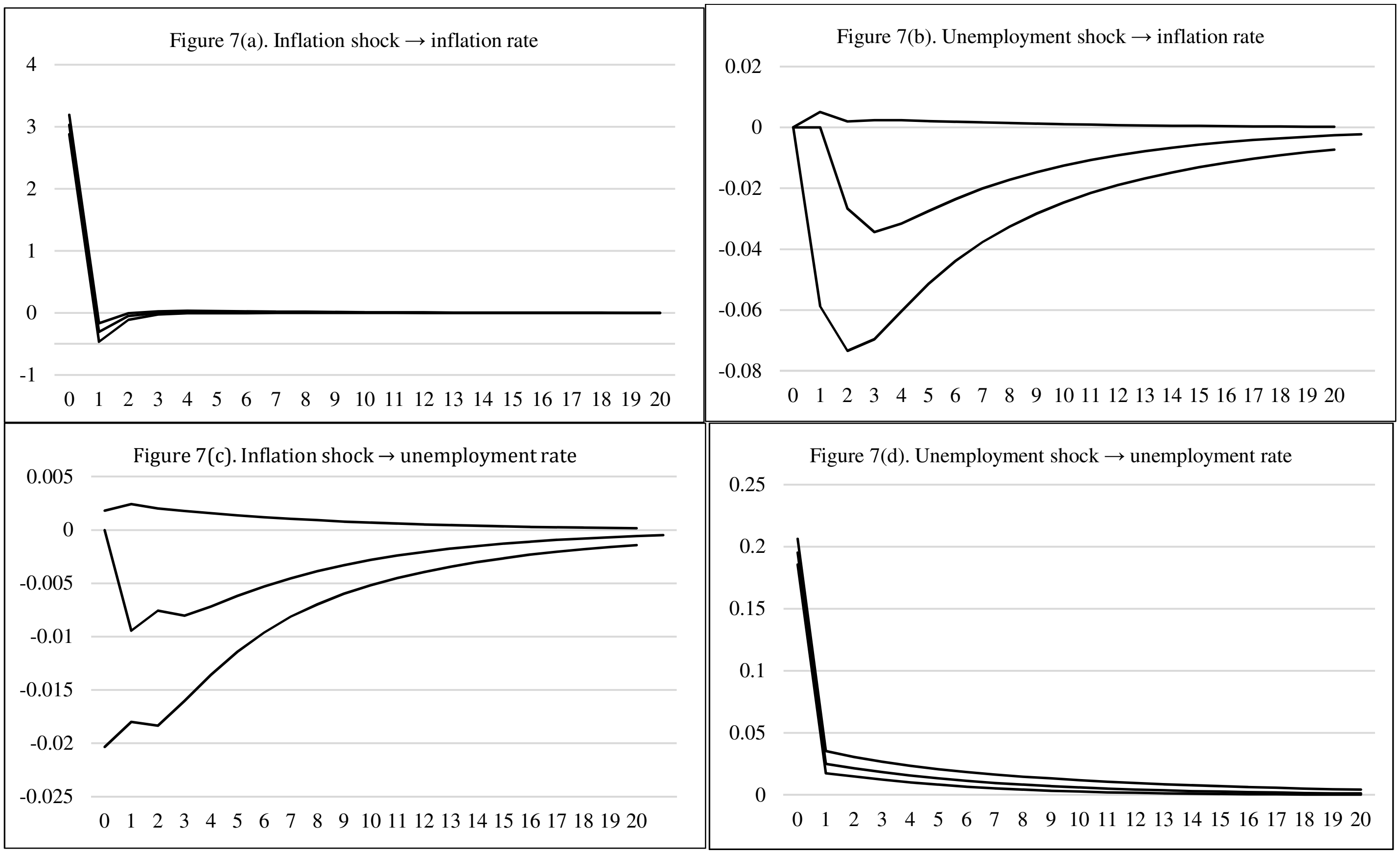


Figure 7. IRF for the QVAR(1) model *Notes*: We present the IRF estimate up to 20 months with the 90% confidence interval.