GESG Seminar, Universidad Francisco Marroquín, 22th July 2016

Dynamic conditional score models of degrees of freedom: filtering with score-driven heavy tails

SZABOLCS BLAZSEK, LUIS ANTONIO MONTEROS

SCHOOL OF BUSINESS, UNIVERSIDAD FRANCISCO MARROQUÍN, GUATEMALA

Motivation

In the body of literature, the most popular measure of financial risk is volatility.

Volatility is the standard deviation of the financial return, conditional on the information available to the investor.

If volatility of a financial asset is high then the probability of extreme observations (large falls or jumps in the price) will be high.

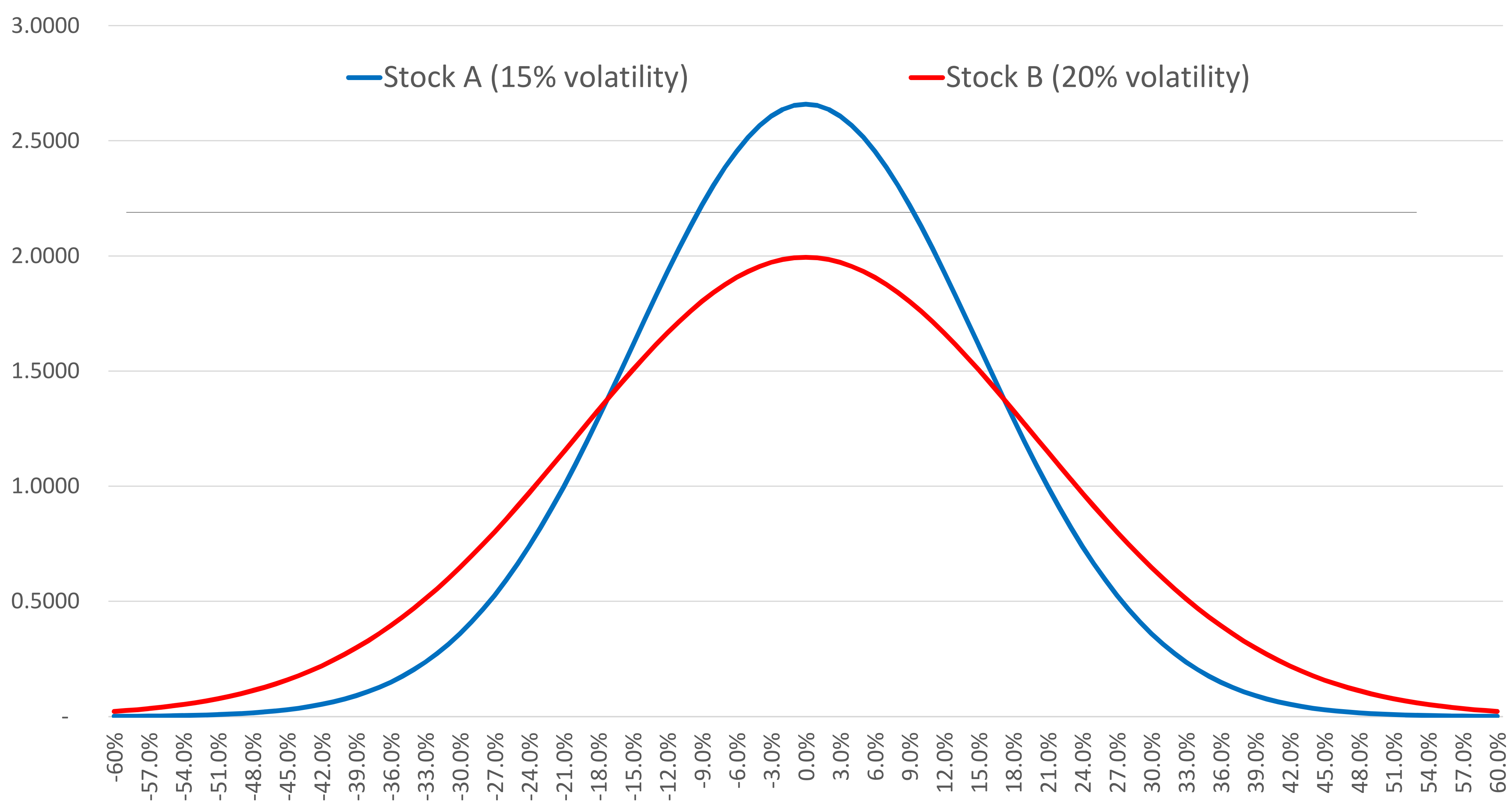
Motivation

A standard volatility model is the **GARCH model** (Bollerslev, 1986; Taylor, 1986). In that model, financial returns are assumed to have **normal distribution** with time-varying standard deviation.

$y_t = \mu_t + \sigma_t \varepsilon_t$ with $\varepsilon_t \sim N(0,1)$

For this model volatility is given by σ_t.

The following figure compares the probabilities of extreme observations for two stocks. In our example, Stock A has 15% annual volatility; Stock B has 20% annual volatility.



3.0000
2.5000
2.0000
1.5000
1.0000
0.5000
-
Stock A (15% volatility)
Stock B (20% volatility)
-60%
-57.0%
-54.0%
-51.0%
-48.0%
-45.0%
-42.0%
-39.0%
-36.0%
-33.0%
-30.0%
-27.0%
-24.0%
-21.0%
-18.0%
-15.0%
-12.0%
-9.0%
-6.0%
-3.0%
0.0%
3.0%
6.0%
9.0%
12.0%
15.0%
18.0%
21.0%
24.0%
27.0%
30.0%
33.0%
36.0%
39.0%
42.0%
45.0%
48.0%
51.0%
54.0%
57.0%
60.0%

Motivation

The last figure shows that for Stock B the probability of extreme observations is higher, hence we may say that Stock B has more financial risk than Stock A.

Motivation

The standard GARCH model with normal distribution may be appropriate to measure and predict financial risk, however when that model is fitted to real data, in many cases, it assigns zero probability to extreme returns.

This may be problematic to investors or risk-managers, since extreme return observations appear time-to-time in practice (their probability is not zero).

Motivation

Motivated by this issue, Bollerslev (1987) suggested another GARCH model for which the probability distribution of the financial return is the **Student's t-distribution**.

Student's t-distribution has an additional parameter with respect to the normal distribution. This (shape) parameter is named as the **degrees of freedom**, and it is denoted by $\exp(\nu)$.

$y_t = \mu_t + \sigma_t \varepsilon_t$ with $\varepsilon_t \sim t[\exp(\nu)]$

For this model volatility is given by $\sigma_t \sqrt{\exp(\nu)/[\exp(\nu) - 2]}$.

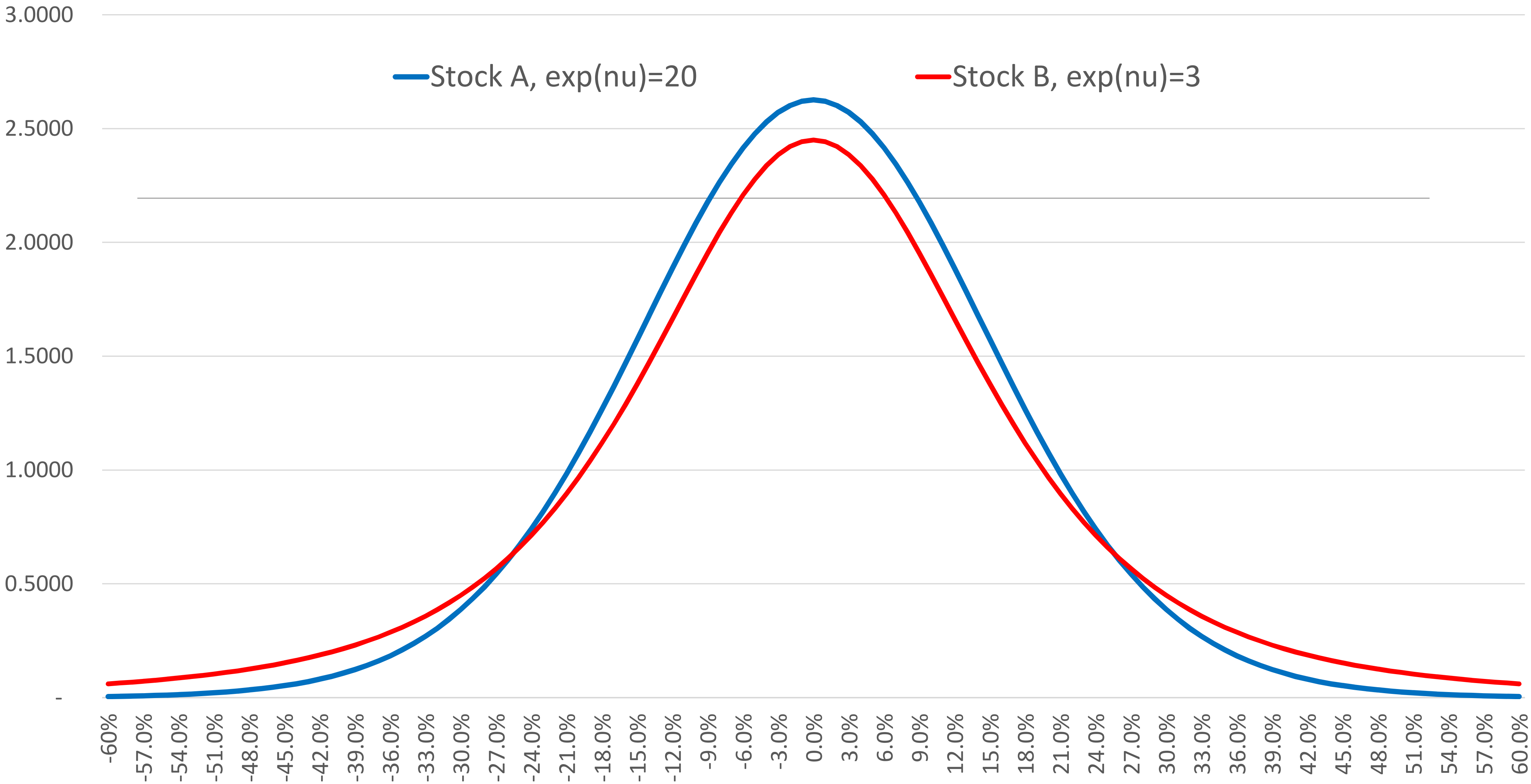
Thus, for this model financial risk is given by a combination of σ_t and ν.

Motivation

The following figure shows what is the effect of the degrees of freedom parameter ν on the probability of extreme financial returns.

For the purpose of illustration, we assume that σ_t is identical for both stocks ($\sigma_t = 15\%$).

We consider $\exp(\nu) = 20$ for Stock A and $\exp(\nu) = 3$ for Stock B.


Stock A, exp(nu)=20
Stock B, exp(nu)=3
3.0000
2.5000
2.0000
1.5000
1.0000
0.5000
-
-60%
-57.0%
-54.0%
-51.0%
-48.0%
-45.0%
-42.0%
-39.0%
-36.0%
-33.0%
-30.0%
-27.0%
-24.0%
-21.0%
-18.0%
-15.0%
-12.0%
-9.0%
-6.0%
-3.0%
0.0%
3.0%
6.0%
9.0%
12.0%
15.0%
18.0%
21.0%
24.0%
27.0%
30.0%
33.0%
36.0%
39.0%
42.0%
45.0%
48.0%
51.0%
54.0%
57.0%
60.0%

Motivation

The last figure shows that the degrees of freedom parameter also influences the probability of extreme financial returns:

A lower value of ν implies a higher probability of extreme returns.

Motivation

In the body of literature, a number of different models of financial risk were suggested.

Those models focus on the time-varying behavior of $\boldsymbol{\sigma_t}$.

The risk models that use the Student's t-distribution typically assume that $\exp(\boldsymbol{\nu})$ is constant over time.

In the present work, we suggest a new model of financial risk for which both σ_t and $\exp(\nu_t)$ are time-varying.

Contribution to the body of literature

Harvey and Chakravarty (2008) suggested the relatively recent **Beta-*t*-EGARCH** model. Beta-*t*-EGARCH is score-driven dynamic volatility model, and it belongs to the family of **dynamic conditional score (DCS)** models (Harvey, 2013).

Beta-*t*-EGARCH uses the Student's t-distribution for the financial return, and assumes that the degrees of freedom parameter $\exp(\nu)$ is constant over time.

We extend Beta-*t*-EGARCH and assume that $\exp(\nu_t)$ is time-varying.

In our model, ν_t is updated by the conditional score. Hence, the extended models belongs to the family of DCS models.

Econometric models

QAR(1) plus Beta-*t*-EGARCH(1,1)

The benchmark model assumes that the degrees of freedom parameter $\exp(\nu)$ is constant over time:

$$y_t = \mu_t + \sigma_t \varepsilon_t = \mu_t + \exp(\lambda_t)\varepsilon_t \text{ with } \varepsilon_t \sim t[\exp(\nu)]$$

Both location μ_t and log-scale λ_t are formulated in a dynamic way, as follows:

$$\mu_t = c + \varphi\mu_{t-1} + \theta u_{\mu,t-1}$$

$$\lambda_t = \omega + \beta\lambda_{t-1} + \alpha u_{\lambda,t-1} + \alpha^* \text{sign}(\mu_t - y_t)(u_{\lambda,t-1} + 1)$$

where $\text{sign}(\cdot)$ is the signum function.

QAR(1) plus Beta-*t*-EGARCH(1,1)

μ_t controls for possible autocorrelation in the expected return.

λ_t controls for possible autocorrelation in volatility. For λ_t, we considers possible asymmetric effects of positive and negative returns on financial risk.

QAR(1) plus Beta-*t*-EGARCH(1,1)

The new information on asset price updates μ_t and λ_t by the terms $u_{\mu,t-1}$ and $u_{\lambda,t-1}$, respectively.

$u_{\mu,t-1}$ and $u_{\lambda,t-1}$ are proportional to the **conditional scores** of y_{t-1}.

The conditional scores for $u_{\mu,t}$ and $u_{\lambda,t}$ are the partial derivatives of the log conditional density of y_t with respect to μ_t and λ_t, respectively.

QAR(1) plus Beta-*t*-EGARCH(1,1)

$u_{\mu,t}$ and $u_{\lambda,t}$ are given by

$$u_{\mu,t} = \frac{\exp(\nu+\lambda_t)\varepsilon_t}{\exp(\nu)+(\varepsilon_t)^2}$$

$$u_{\lambda,t} = \frac{[\exp(\nu)+1](\varepsilon_t)^2}{\exp(\nu)+(\varepsilon_t)^2} - 1$$

QAR(1) plus Beta-*t*-EGARCH(1,1) with time-varying degrees of freedom

Extended model:

$y_t = \mu_t + \sigma_t \varepsilon_t = \mu_t + \exp(\lambda_t)\varepsilon_t$ with $\varepsilon_t \sim t[\exp(\nu_t)]$

μ_t and σ_t are formulated in the same way as for the benchmark model:

$\mu_t = c + \varphi \mu_{t-1} + \theta u_{\mu,t-1}$

$\lambda_t = \omega + \beta \lambda_{t-1} + \alpha u_{\lambda,t-1} + \alpha^* \text{sign}(\mu_t - y_t)(u_{\lambda,t-1} + 1)$

For $u_{\mu,t}$ and $u_{\lambda,t}$, the ν term is replaced by ν_t.

QAR(1) plus Beta-*t*-EGARCH(1,1) with time-varying degrees of freedom

Extended model:

The log of the degrees of freedom is modelled as:

$$\nu_t = \delta + \gamma \nu_{t-1} + \kappa u_{\nu,t-1}$$

where $u_{\nu,t}$ is the conditional score with respect to ν_t.

QAR(1) plus Beta-*t*-EGARCH(1,1) with time-varying degrees of freedom

Conditional score:

$$u_{\nu,t} = \frac{\partial \ln f(y_t|y_1,\ldots,y_{t-1})}{\partial \nu_t} = \frac{1}{2}\exp(\nu_t)\Psi^{(0)}\left[\frac{\exp(\nu_t)+1}{2}\right]$$

$$-\frac{1}{2}\exp(\nu_t)\Psi^{(0)}\left[\frac{\exp(\nu_t)}{2}\right] - \frac{1}{2}$$

$$+\frac{\epsilon_t^2[\exp(\nu_t)+1] - \exp(\nu_t)[\epsilon_t^2+\exp(\nu_t)]\ln[\epsilon_t^2\exp(-\nu_t)+1]}{2[\epsilon_t^2+\exp(\nu_t)]}$$

QAR(1) plus Beta-*t*-EGARCH(1,1) with time-varying degrees of freedom

In the previous formula $\Psi^{(0)}(\cdot)$ is the digamma function (also named as polygamma function of order zero).

(The digamma function is the derivative of the log of the gamma function.)

QAR(1) plus Beta-t-EGARCH(1,1) with time-varying degrees of freedom

The following figure presents $u_{\mu,t}$, $u_{\lambda,t}$ and $u_{\nu,t}$, as a function of ε_t (the new information arrives to the market):

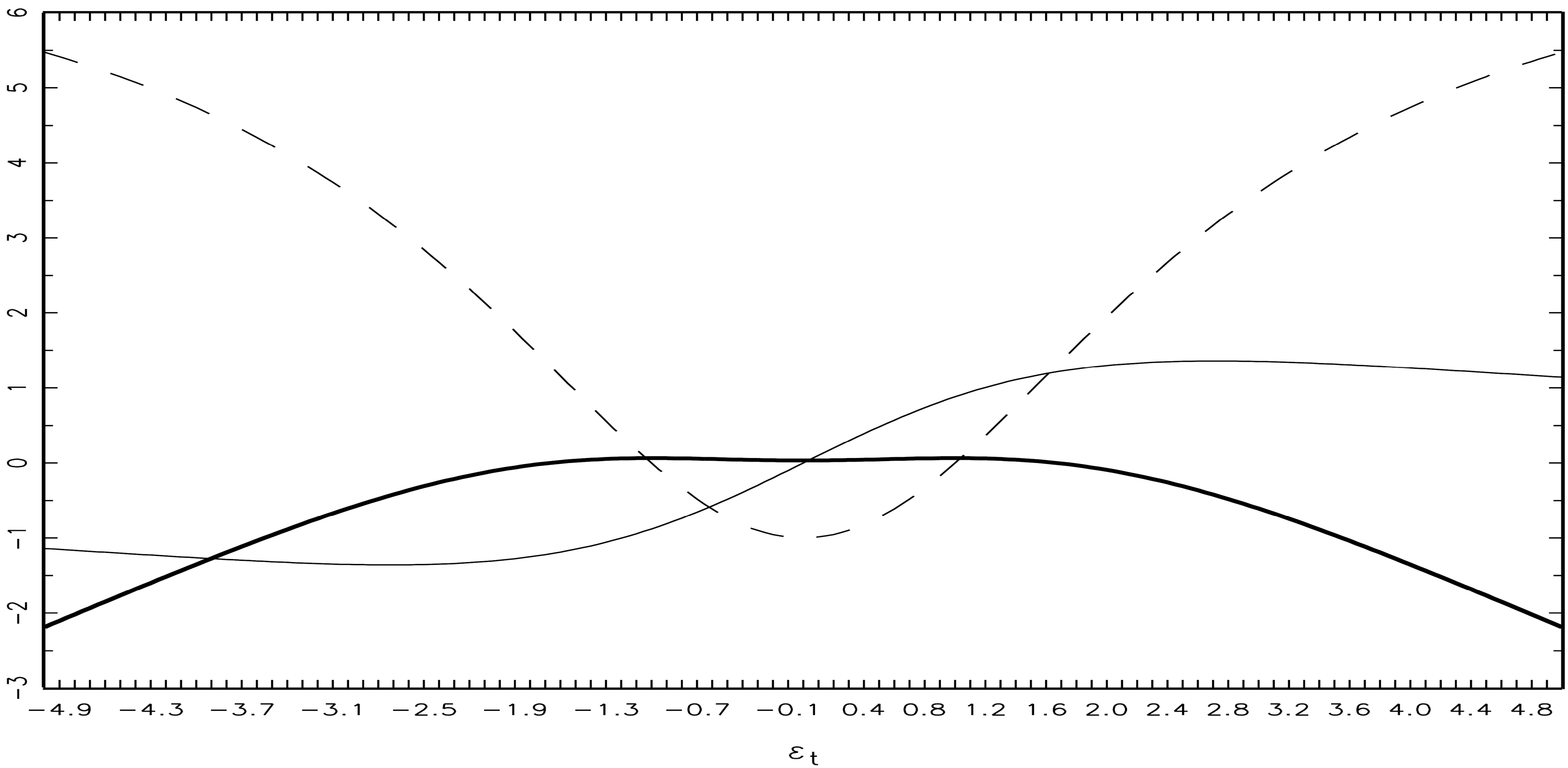


Fig. 1. Score functions for DCS location, scale and degrees of freedom

Notes: $u_{\mu,t}$ (solid thin); $u_{\lambda,t}$ (dashed thin); $u_{\nu,t}$ (solid thick)

Data

Data

Daily returns on the S&P 500 index for period 4th January 1950 to 12 May 2016 ($T = 16{,}697$ observations).

Daily returns for a random sample of 150 components of the S&P 500. For each firm the entire historical time period available is used.

Those 150 firms are from different industries of the US economy: *information technology; health care; consumer staples; consumer discretionary; financials; energy; materials; industrials; utilities*.

Data

Some well-known firms of the random sample:

Apple; Citigroup; Chevron; Walt Disney; FedEx; Twenty-First Century Fox; General Motors; Goldman Sachs; Hasbro; Harley-Davidson; Hewlett-Packard; IBM; Johnson & Johnson; Eli Lilly; Moody's; 3M; NIKE; Northrop Grumman; Oracle; Pepsico; AT&T; Wells Fargo

Empirical results

Empirical results

(A) In-sample statistical performance

- S&P 500
- Random sample of 150 components of the S&P 500

(B) Out-of-sample density forecast performance for S&P 500

(C) Monte Carlo Value-at-Risk (VaR) application for S&P 500

(A) In-sample statistical performance S&P 500

We compare the statistical performance of QAR plus Beta-*t*-EGARCH with constant and time-varying degrees of freedom.

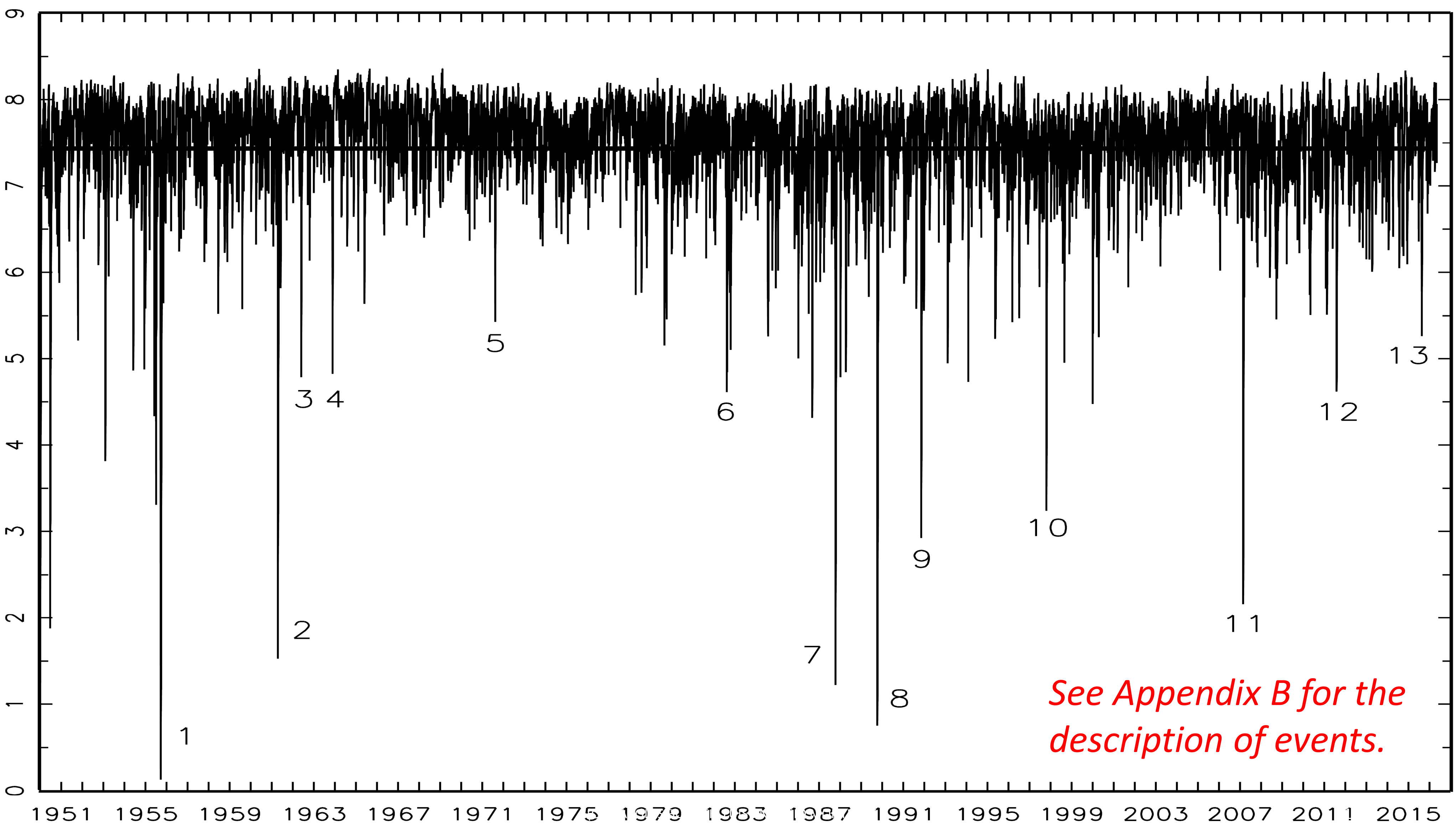
We use the *Likelihood-Ratio (LR) test*, *Akaike Information Criterion (AIC)*, *Bayesian Information Criterion (BIC)* and *Hannan-Quinn Criterion (HQC)*.

All these metrics suggest that the statistical performance of the extended DCS model is superior to the benchmark model, for the S&P 500.

(A) In-sample statistical performance S&P 500

We find significant dynamics in the degrees of freedom parameter for the S&P 500: $\gamma = 0.8260$.

The following figure presents the evolution of the **degrees of freedom parameter** for the extended DCS model:



9
8
7
6
5
4
3
2
1
0
1951
1955
1959
1963
1967
1971
1975
1979
1983
1987
1991
1995
1999
2003
2007
2011
2015
1
2
3 4
5
6
7
8
9
10
11
12
13
See Appendix B for the description of events.

(A) In-sample statistical performance 150 components

We compare the statistical performance of the benchmark and extended models by using the LR test. We find that for 59% of the 150 firms, the extended DCS model has superior statistical performance that is significant at the 10% level of significance.

The results suggest that the extended DCS model is superior for those industries, which produce products that people usually are unwilling to cut out of their budgets regardless of their financial situation:

(A) In-sample statistical performance 150 components

For the **consumer staples**, **financials**, **health care** and **utilities** sectors for 92%, 70%, 75% and 100% of the firms within the sector, respectively, the extended DCS model dominates at the 10% level of significance.

For the **consumer discretionary**, **energy**, **industrials**, **information technology**, **materials** and **telecommunications services** sectors for 46%, 57%, 41%, 37%, 50% and 50% of the firms within the sector, respectively, the two models are identical.

(B) Out-of-sample density forecast performance for S&P 500

Estimation window: 4th January 1950 to 19th October 1987

Forecast window: 20th October 1987 to 12th May 2016

We use the *Amisano-Giacomini (2007) out-of-sample density forecast comparison test*.

We compare the predictive performance of benchmark and extended models.

We find that the extended DCS model is superior to the benchmark model, at the 5% level of significance.

(C) Monte Carlo Value-at-Risk (VaR) application for S&P 500

We apply the benchmark and extended DCS models to estimate the *Monte Carlo VaR* of a long position in the S&P 500 (Jorion, 2006).

The position is opened on the day after the last day of the full data window (13th May 2016).

For Monte Carlo VaR, we simulate 10,000 trajectories of S&P 500.

We consider the 5-days, 10-days and 20-days holding periods for the long position.

We consider the 90%, 95%, 97.5% and 99% confidence levels of VaR.

We also estimate historical VaR and delta-normal VaR values.

Thank you for your attention!

SBLAZSEK@UFM.EDU, LUISMONTEROS@UFM.EDU

Amisano, G. and Giacomini, R. (2007) Comparing density forecasts via weighted likelihood ratio tests, *Journal of Business and Economic Statis*tics, 25, 177-190.

Bollerslev, T. (1986) Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics*, 31, 307-327.

Bollerslev, T. (1987) A conditionally heteroskedastic time series model for security prices and rates of return data}, *Review of Economics and Statistics*, 59, 542-547.

Harvey, A. C. (2013) *Dynamic Models for Volatility and Heavy Tails*, Cambridge University Press, Cambridge, UK.

Harvey, A. C. and Chakravarty, T. (2008) Beta-t-(E)GARCH, Cambridge Working Papers in Economics 0840, Faculty of Economics, University of Cambridge, Cambridge, UK.

Jorion, P. (2006) *Value at Risk: The New Benchmark for Managing Financial Risk*, McGraw-Hill, New York.

Taylor, S. (1986) *Modelling Financial Time Series*, Wiley, Chichester.