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Szabolcs Blazsek* **and Richard Bowen**

Score-driven cryptocurrency and equity portfolios

Abstract:

Motivated by the recent start of BITO which replicates Bitcoin, we study whether BITO can improve equity portfolios. Investors may prefer BITO to Bitcoin because: (i) BITO is traded on a regulated market, while cryptocurrency exchanges are largely unregulated; (ii) the fee for BITO is lower than the fees at cryptocurrency exchanges. For the first time in the literature, we use score-driven models to optimize equity-BITO portfolios, in order to show that BITO improves equity portfolio performances. For the equity component we use VOO to replicate the Standard & Poor's 500 (S&P 500) index. The full sample period is from September 18, 2014 to January 21, 2022. We compare the performances of VOO, Bitcoin, 40 portfolios for DCC (dynamic conditional correlation) models, and 900 portfolios for score-driven copulas. Two investment periods are considered: (i) full investment period (March 21, 2018 to January 21, 2022); (ii) COVID-19 (coronavirus pandemic of 2019) investment period (February 24, 2020 to January 21, 2022). We find that the performance of VOO is improved by the score-driven portfolios for both investment periods, the improvement is more significant for the COVID-19 investment period, and several score-driven portfolios are superior to the DCC portfolios.

Keywords: equity-cryptocurrency portfolios, COVID-19 (coronavirus pandemic of 2019), dynamic conditional score (DCS) models, generalized autoregressive score (GAS) models, score-driven copulas

JEL classification codes: C32, C52, C58, G11

* **Corresponding author: Szabolcs Blazsek,** School of Business, Universidad Francisco Marroquín, Calle Manuel F. Ayau (6 Calle final), zona 10, Guatemala City 01010, Guatemala, E-mail: sblazsek@ufm.edu

Richard Bowen, School of Business, Universidad Francisco Marroquín, Guatemala City, E-mail: richardbowen@ufm.edu

1 Introduction

On October 19, 2021, the trading of the first Bitcoin exchange traded fund (ETF) (ticker: BITO) began at New York Stock Exchange (NYSE) Arca. Although Bitcoin was invented in 2008 and its trading started shortly after, due to its largely unregulated cryptocurrency markets, it took several years for the Securities and Exchange Commission (SEC) to approve an ETF that replicates the price of Bitcoin. Compared to investing in Bitcoin, one of the advantages of investing in BITO is its significantly lower transaction fee. For example, on the Coinbase cryptocurrency exchange platform, opening a position in Bitcoin with a value under $10,000 has a fee of 0.5% of the position value. The fees are also significant at other cryptocurrency exchanges (e.g. Binance, PrimeXBT, crypto.com, among others). On the other hand, for BITO, the trading fee is represented by the 0.95% annual expense ratio of the ETF, which is significantly lower than the fees of cryptocurrency exchanges. This may motivate many investors to choose BITO instead of Bitcoin for cryptocurrency investments. Moreover, many equity investors who have not included cryptocurrencies in equity portfolios yet, may include BITO in their portfolios in the future by trading on regulated markets and using the same trading platforms.

We study whether cryptocurrencies improve equity portfolios. We consider the following cryptocurrencies: Bitcoin (BTC-USD), Ethereum (ETH-USD), Ripple (XRPUSD), Binance Coin (BNBUSD), Cardano (ADA-USD), Solana (SOL-USD), Litecoin (LTC-USD), Dash (DASH-USD), and Monero (XMR-USD). We focus on Bitcoin because the other cryptocurrencies in the list either have sample periods that are too short for out-of-sample portfolio performance evaluation, or they are not among the most important 10 cryptocurrencies according to market capitalization at the date of this paper.

Our objective is to perform a portfolio performance study for investors who consider adding BITO to equity portfolios. Since for BITO only a very short time series is available currently, we decided to use data on Bitcoin prices (which are replicated by BITO since October 2021), and we subtract from Bitcoin returns the annual expense ratio of BITO. In this way, we get a longer time series representing BITO performance for the period of September 18, 2014 to January 22, 2022.

For the equity component of the portfolio, we use the Vanguard S&P 500 ETF (ticker: VOO), which replicates the Standard & Poor's 500 (S&P 500) index. For VOO, the available observation period is from September 10, 2010 to January 21, 2022. For the VOO return calculation, we subtract from VOO returns the 0.03% annual expense ratio of the ETF. We splice the Bitcoin and VOO series, hence the full sample period in this paper is from September 18, 2014 to January 21, 2022. We note that VOO is traded from Monday to Friday, while Bitcoin is traded seven days a week. For the spliced dataset, we use the opening prices from Monday to Friday for both ETFs.

The main contribution of this paper is that for the optimization of VOO-Bitcoin portfolios we use score-driven models, which to our knowledge has never been considered in the literature. Score-driven models are introduced in the works of Creal, Koopman, and Lucas (2008, 2011, 2013), Harvey and Chakravarty (2008), and Harvey (2013). Creal, Koopman, and Lucas (2013) and Harvey (2013) name those models generalized autoregressive score (GAS) and dynamic conditional score (DCS) models, respectively. Those models are observation-driven models (Cox 1981), for which the dynamic parameters are updated by the scaled partial derivatives of the log conditional density of the dependent variables with respect to dynamic parameters (hereinafter, the updating terms are named scaled score functions).

Some of the advantages of the score-driven models over the classical observation-driven models are: (i) Score-driven models are robust to outliers and missing data (Harvey 2013). (ii) Several score-driven models are generalizations of classical observation-driven models (Creal, Koopman, and Lucas 2013; Harvey 2013). (iii) Blasques, Koopman, and Lucas (2015) show that a score-driven update locally reduces the Kullback–Leibler divergence in expectation at every step, and only the score-driven updates have this property. These advantages may motivate the use of score-driven models for the estimation of portfolio mean and volatility. In relation to this, we refer to the works of Atskanov (2016) and Bernardi and Catania (2018), which use score-driven models for the optimization of equity portfolios (i.e. those authors do not consider cryptocurrencies in the portfolios).

In the literature on score-driven models some papers use data on cryptocurrencies, but to the best of our knowledge none of those papers investigate the performances of portfolios including cryptocurrencies. The following works of the literature study the volatility, density, or value-at-risk (VaR) forecasting performance of score-driven models for cryptocurrencies: Catania, Grassi, and Ravazzolo (2018), Troster et al. (2019), Ranjbar (2020), Catania and Grassi (2021), Jeribi and Ghorbel (2021), and Jiang et al. (2022). These works find that the volatility, density, or VaR forecasting performance of score-driven models are superior to those of classical volatility models (Engle 1982; Bollerslev 1986; Nelson 1991), motivating the score-driven portfolio analysis of the present paper. In addition, we refer to the work of Matkovskyy, Jalan, and Dowling (2020), in which the effects of economic policy uncertainty on the relationship between Bitcoin and equity markets are studied. We also refer to the recent work of Yarovaya, Matkovsyy, and Jalan (2021), in which the authors study the herding behavior of cryptocurrencies for the period of the COVID-19 pandemic, without studying portfolio performances.

We study the performance of VOO-Bitcoin portfolios for the full investment period of March 21, 2018 to January 21, 2022, and the COVID-19 (coronavirus pandemic of 2019) investment period of February 24, 2020 to January 21, 2022. Portfolio expected return forecasting and portfolio volatility forecasting are performed by using a rolling-window estimation and forecasting approach. We compare the performances of 900 portfolios for score-driven copulas, 40 portfolios for DCC (dynamic conditional correlation) models (Engle 2002), and the benchmark portfolio which only includes VOO. We study whether the performance of score-driven portfolios are better or worse than VOO.

We use five score-driven copulas (Clayton; rotated Clayton; Gumbel; rotated Gumbel; Student's t). Score-driven copulas are used in several works, for example: Boudt et al. (2012); Avdulaj and Barunik (2013, 2015); Creal, Koopman, and Lucas (2013); De Lira Salvatierra and Patton (2015); Koopman, Lit, and Lucas (2015); Atskanov (2016); Bartels and Ziegelmann (2016); Harvey and Thiele (2016); Koopman, Lucas, and Scharth (2016); Oh and Patton (2016); Cerrato et al. (2017); Ayala and Blazsek (2018a, 2018b); Bernardi and Catania (2018). We use four portfolio optimization strategies (minimum-variance; Sharpe ratio-based mean-variance; two utility function-based mean-variance). We use five portfolio weight updating frequencies (weekly; monthly; quarterly; semi-annual; annual). We use nine combinations of (i) AR (autoregressive)-t-GARCH (generalized autoregressive conditional heteroskedasticity) (Box and Jenkins 1970; Bollerlsev 1987), (ii) QAR (quasi-AR)-Beta-t-EGARCH (exponential GARCH) (Harvey 2013; Harvey and Chakravarty 2008), (iii) QAR-Beta-Gen-t-EGARCH (generalized t-distribution) (Harvey and Lange 2017). By using the aforementioned four portfolio optimization strategies and five portfolio weight updating frequencies, we also study the performances of AR-GARCH-DCC portfolios for the normal and t distributions.

We find that for both the full and COVID investment periods, the VOO performance is significantly improved by the score-driven portfolios, which are also superior to the DCC portfolios.

In the remainder of this paper, Section 2 reviews the literature, Section 3 presents the econometric methods, Section 4 describes the data, Section 5 summarizes the results, and Section 6 concludes.

2 Literature review

In this section, we review the literature on score-driven models for cryptocurrencies. The most closely related works to our paper investigate the volatility, density, or VaR forecasting performances of score-driven models for cryptocurrencies. It is important to note that, to our knowledge, none of the papers in the literature on score-driven models study portfolio performances for cryptocurrencies.

The work of Catania, Grassi, and Ravazzolo (2018) focuses on predicting the volatilities of Bitcoin, Ethereum, Litecoin, and Ripple. For volatility modeling, the authors use the GARCH model (M1), and a score-driven volatility model for the generalized hyperbolic skewed Student's t-distribution (M2). The following extensions of the score-driven volatility model are also used: leverage effects (M3), time-varying skewness (M4), and fractional integration (M5). We note that the work of Catania and Grassi (2017) develops (M2) to (M4). Daily data for Bitcoin and Litecoin are used for the period of April 29, 2013 to December 1, 2017. Daily data for Ethereum are used for the period of August 8, 2013 to December 1, 2017. Daily data for Ripple are used for the period of August 5, 2013 to December 1, 2017. The results indicate that (M2) is superior to (M1), (M3), (M4), and (M5).

In the work of Troster et al. (2019), Bitcoin data are used for the period of July 19, 2010 to April 16, 2019. For the classical model, the authors use an AR(1) expected return specification with GARCH, EGARCH, APARCH (asymmetric power ARCH), GARCH with leverage effects, T-GARCH, CGARCH, NGARCH, and H-GARCH volatility specifications for the normal, Student's t, skewed t, Johnson's S_U, and general error distributions (Troster et al. 2019). For the score-driven volatility models, the normal, Student's t, skewed t, and asymmetric t distributions are used. Out-of-sample volatility forecasting performances and VaR backtests of the different volatility models are compared, and the authors find that the score-driven models are superior to the classical models.

In the work of Matkovskyy, Jalan, and Dowling (2020), the effects of economic policy uncertainty on the relationship between Bitcoin and equity markets are studied. The Bitcoin markets considered are in the United States (US), the United Kingdom (UK), Europe, and Japan. The authors use daily data for the period of April 27, 2015 to October 25, 2018. The stock indices considered are NASDAQ 100, S&P 500, Euronext 100, FTSE 100, and NIKKEI 225. Economic policy uncertainty variables are measured for the US, the UK, Europe, and Japan. The score-driven volatility and copula models are for the Student's t-distribution. Impulse responses of Bitcoin volatility and Bitcoin-equity correlations to economic policy shocks are estimated. The findings show that Bitcoin can be used as a hedging tool against equity markets, in the case of US economic policy uncertainty.

The work of Ranjbar (2020) focuses on the in-sample and out-of-sample performances of the

GARCH, realized-GARCH, GARCH with leverage effects, EGARCH, and score-driven volatility models for Bitcoin. That author uses data from the following Bitcoin markets: Coinbase, Bitfinex, and Bitstamp; normal distribution for all classical univariate volatility models; the normal and Student's t distributions for the score-driven volatility models; the multivariate normal distribution for the multivariate classical volatility model; and the multivariate t-distribution for the multivariate score-driven model (refer to Creal, Koopman, and Lucas 2011 for the latter).

The work of Catania and Grassi (2021) uses score-driven volatility models, which specify leverage effects, long memory (captured by a two-component volatility filter; see Harvey 2013), and time-varying skewness and kurtosis (time-varying kurtosis extends the works of Catania and Grassi 2017 and Catania, Grassi, and Ravazzolo 2018), to forecast the volatility, density, and quantile (i.e. VaR and ES, expected shortfall) for 606 cryptocurrencies. Those authors use daily data on the following cryptocurrencies having the largest market capitalization: Bitcoin, Ethereum, Ripple, and Litecoin, for which there are at least 700 observations. For the score-driven models, they use the Beta-Skew-t-EGARCH model (i.e. skewed t-distribution) and the generalized hyperbolic skewed Student's t-distribution. They also highlight the importance of the time-varying skewness in the predictions.

In the work of Jeribi and Ghorbel (2021), the authors use daily data on gold, cryptocurrency, and equity returns. The equity returns are from stock indices of Brazil, Russia, India, China, and South Africa. The cryptocurrencies considered are Bitcoin, Dash, Ethereum, Monero, and Ripple. For the equity indices from developed countries the S&P 500, NASDAQ, FTSE, and NIKKEI are used. The score-driven model for the gold, cryptocurrency, and equity uses the normal, Student's t, and skewed t distributions. For the classical model, those authors use a multivariate GARCH model, named GO-GARCH, with a non-zero conditional mean return component. They also use data for the period of January 1, 2016 to December 31, 2019. Further, in the empirical analysis, VaR backtesting is performed and the time-varying correlation of cryptocurrencies with equity indices is analyzed.

In the work of Yarovaya, Matkovskyy, and Jalan (2021), those authors use hourly data for the period of 0:00 a.m. January 1, 2019 to 8:00 p.m. March 13, 2020, and they study the herding behavior of cryptocurrencies for the period of the COVID-19 pandemic. They also study the question: Does the COVID-19 pandemic amplify herding behavior in cryptocurrency markets? The cryptocurrencies used are Bitcoin, Ethereum, and Litecoin. Those authors study price data for these cryptocurrencies for the top four cryptocurrency markets by trading volume: USD (United States dollar), EUR (euro), JPY (Japanese yen), and KWR (South Korean won); aggregate the Bitcoin, Ethereum, and Litecoin returns for each market (i.e. USD, EUR, JPY, or KWR), by using the average Bitcoin, Ethereum, and Litecoin return for each hour; and estimate time-varying correlations of cryptocurrency returns by using a score-driven model for the multivariate t-distribution. For all cryptocurrency markets, the results suggest that COVID-19 does not amplify herding in cryptocurrency markets.

In the work of Jiang et al. (2022), the accelerating (aGAS) technique for a Gaussian-Cauchy mixture model is introduced, which is applied to forecasting VaR for Bitcoin, Ripple, and Litecoin. Those authors use daily data from the start of Bitcoin, Ripple, and Litecoin to June 15, 2020 (i.e. part of the COVID-19 period is included in the sample), and find that, compared to the classical AR moving average (ARMA)-GARCH model with Gaussian, skewed t, and nonparametric innovations, the proposed aGAS model for a Gaussian-Cauchy mixture performs well in out-of-sample VaR forecasting.

3 Methods

The daily returns of the assets in the portfolio are $r_{k,t} = (p_{k,t} - p_{k,t-1})/p_{k,t-1}$ for days $t = 1, \ldots, T$, where $k = 1, 2$ indicate VOO and Bitcoin, respectively. The variable $p_{k,t}$ is the opening price, and we use pre-sample data for $p_{k,0}$. We assume that the risk-free rate is zero, hence $r_{k,t}$ also represents the excess return over the risk-free rate. The portfolio excess return is $r_{P,t} = w'r_t = w_1 r_{1,t} + w_2 r_{2,t}$, where $r_t \equiv (r_{1,t}, r_{2,t})'$ is a vector of portfolio excess returns and $w \equiv (w_1, w_2)'$ is a vector of portfolio weights.

The first investment strategy is 100% investment in VOO, which is the benchmark portfolio in this paper. For the alternative strategies, the optimal portfolio weights are chosen as follows: (i) Minimizing the portfolio variance $\sigma_P^2 = w'\Sigma w$, where Σ is the variance-covariance matrix of the excess returns of all assets in the portfolio. (ii) Maximizing the Sharpe ratio μ_P/σ_P, where $\mu_P = w'\mu$ is the expected excess portfolio return, where μ is a column vector of expected excess returns of all assets in the portfolio, and σ_P is the volatility of the excess portfolio return. (iii) Maximizing the utility function $\mu_P - (\zeta/2)\sigma_P^2$ by using the alternative risk aversion coefficients: $\zeta = 1$ and $\zeta = 4$ (DeMiguel, Garlappi, and Uppal 2009). For strategies (i)-(iii), we consider weekly, monthly, quarterly, semi-annual, and annual alternative updates of optimal portfolio weights. In the portfolio return application case study of the present paper, we assume that the investor takes long positions in both VOO and Bitcoin.

In the following, we present the econometric models that we use for the estimation of the expected excess returns μ and the variance-covariance matrix of excess returns Σ.

3.1 Classical models

First, for the marginal distribution we use the AR(1) (Box and Jenkins 1970) plus GARCH(1,1) with leverage effects (Black 1976; Bollerslev 1987; Glosten, Jagannathan, and Runkle 1993) model:

$$r_{k,t} = \mu_{k,t} + v_{k,t} = \mu_{k,t} + \lambda_{k,t}^{1/2}\epsilon_{k,t} \tag{1}$$

$$\mu_{k,t} = c_k + \phi_k r_{k,t-1} = c_k + \phi_k(\mu_{k,t-1} + v_{k,t-1}) \tag{2}$$

$$\lambda_{k,t} = \omega_k + \beta_k\lambda_{k,t-1} + [\alpha_k + \alpha_k^{*}\mathbb{1}(v_{k,t-1} < 0)]v_{k,t-1}^2 \tag{3}$$

for $k = 1, 2$, where the excess return $r_{k,t}$ is the sum of the expected excess return $\mu_{k,t} = E(r_{k,t}|\mathcal{F}_{t-1};\Theta)$ and the unexpected excess return $v_{k,t}$, where $\mathcal{F}_{t-1} = \sigma(r_{k,1},\ldots,r_{k,t-1} : k = 1, 2)$, and Θ is the vector of the time-invariant parameters. The parameters c_k, ϕ_k, θ_k are real numbers. The parameters $\omega_k > 0$, $\beta_k > 0$, $\alpha_k > 0$, and $\alpha_k^{*} + \alpha_k > 0$. The unexpected excess return is the product of the dynamic scale parameter $\lambda_{k,t}^{1/2}$ and the i.i.d. error term. For the standardized error term, we consider the alternatives $\epsilon_{k,t} \sim N(0, 1)$ and $\epsilon_{k,t} \sim t(\nu_k)$ with the Student's t-distribution. The conditional standard deviation of the unexpected excess return (i.e. conditional volatility) for the Gaussian distribution is $\sigma_{k,t} = \mathrm{SD}(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \lambda_{k,t}^{1/2}$, and for the t-distribution is $\sigma_{k,t} = \mathrm{SD}(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \lambda_{k,t}^{1/2}[\nu_k/(\nu_k - 2)]^{1/2}$.

For the filter $\lambda_{k,t}$, we consider the possibility of leverage effects α_k^{*} for $k = 1, 2$, for which negative unexpected excess returns are identified by using the indicator function $\mathbb{1}(x)$. The filter $\mu_{k,t}$ is initialized by $c_k/(1 - \phi_k)$, and it is covariance stationary if $|\phi_k| < 1$. The filter $\lambda_{k,t}$ for $k = 1, 2$ is initialized by parameters $\lambda_{1,1}$ and $\lambda_{2,1}$, respectively, and it is covariance stationary if $\beta_k + \alpha_k + \alpha_k^{*}/2 < 1$.

Second, for the correlation coefficients we use the DCC model. For the estimation of the AR-GARCH-DCC models, we use the two-step maximum likelihood (ML) method (Engle 2002).

3.2 Score-driven models

First, for the marginal distribution we use three alternatives. The first one is the classical AR plus t-GARCH model from Section 3.1. For the other alternatives we use the following score-driven models:

$$r_{k,t} = \mu_{k,t} + v_{k,t} = \mu_{k,t} + \exp(\lambda_{k,t})\epsilon_{k,t} \tag{4}$$

$$\mu_{k,t} = c_k + \phi_k \mu_{k,t-1} + \theta_k s_{\mu,k,t-1} \tag{5}$$

$$\lambda_{k,t} \quad = \omega_k + \beta_k \lambda_{k,t-1} + \alpha_k s_{\lambda,k,t-1} + \alpha_k^* \text{sgn}(-v_{k,t-1})(s_{\lambda,k,t-1} + 1) \tag{6}$$

for $k = 1, 2$, where $\exp(\cdot)$ is the exponential function, and for the $\epsilon_{k,t}$ i.i.d. error term we use the following five alternative distributions: $\epsilon_{k,t} \sim t[0, 1, \exp(\nu_k) + 2]$, $\epsilon_{k,t} \sim \text{Gen-}t[0, 1, \exp(\nu_k) + 2, \exp(\eta_k)]$, where $\nu_k \in \mathbb{R}$ and $\eta_k \in \mathbb{R}$ (Harvey and Chakravarty 2008; Harvey 2013; Harvey and Lange 2017).

The updating terms of Eqs. (5) and (6) are the scaled score function $s_{\mu,k,t}$, and the score function $s_{\lambda,k,t}$, respectively, which we define later. For $\lambda_{k,t}$, we consider the possibility of leverage effects α_k^* for $k = 1, 2$, for which asymmetry is measured using the signum function $\text{sgn}(\cdot)$. We note that we do not use the skewed generalized t-distribution, which is more general than the Student's t and Gen-t distributions, because Eq. (6) captures asymmetries by using the leverage effects term.

In the literature, the sigma-algebra $\mathcal{F}_{t-1}$ includes the initial values of all score-driven filters (e.g. Blasques et al. 2021). In the present paper, we use the same sigma-algebra $\mathcal{F}_{t-1}$ for the score-driven models and for the classical models (Section 3.1), because the score-driven filters are initialized by using some elements of Θ; i.e. $\mu_{k,t}$ for $k = 1, 2$ are initialized by $c_k/(1 - \phi_k)$ for $k = 1, 2$, respectively, and $\lambda_{k,t}$ for $k = 1, 2$ are initialized by the parameters $\lambda_{k,1}$ for $k = 1, 2$, respectively.

For each score-driven probability distribution, the log conditional density of $r_{k,t}|(\mathcal{F}_{t-1}; \Theta)$, the scaled score function $s_{\mu,k,t}$, the score function $s_{\lambda,k,t}$, the conditional expected return $E(r_{k,t}|\mathcal{F}_{t-1}; \Theta)$, and the conditional volatility $\sigma(r_{k,t}|\mathcal{F}_{t-1}; \Theta)$ are presented in Appendix A. For the results presented in Appendix A, we refer to the works of Harvey and Chakravarty (2008), Harvey (2013), Caivano and Harvey (2014), Blazsek, Ho, and Liu (2018), and Ayala, Blazsek, and Escribano (2019).

We use all possible combinations of the AR-t-GARCH, QAR-Beta-t-EGARCH, and QAR-Beta-Gen-t-EGARCH models which provide nine alternatives of the marginal models. The transformations of the error term by the scaled score function $s_{\mu,k,t}$ and the score function $s_{\lambda,k,t}$ within QAR-Beta-t-EGARCH, and QAR-Beta-Gen-t-EGARCH are presented in Appendix A.

Second, for the dynamic association of VOO and Bitcoin excess returns, we use the score-driven Clayton, rotated Clayton, Gumbel, rotated Gumbel, and Student's t copulas. For all copulas, we model the score-driven parameter of association ρ_t as follows: (i) We use the dynamic parameter $\tilde{\rho}_t$:

$$\tilde{\rho}_t = \delta + \gamma \tilde{\rho}_{t-1} + \kappa s_{\rho,t-1} \tag{7}$$

where the conditional copula score is given by $s_{\rho,t} = \partial \ln c_t[F_1(r_{1,t}|\mathcal{F}_{t-1};\Theta), F_2(r_{2,t}|\mathcal{F}_{t-1};\Theta)]/\partial\rho_t$, and where $F_1(\cdot|\cdot)$ and $F_2(\cdot|\cdot)$ are the marginal conditional distribution functions of VOO and Bitcoin excess returns, respectively. For each copula, the copula density function c_t and the copula score $s_{\rho,t}$ are presented in Appendix B. We initialize $\tilde{\rho}_t$ by using $\delta/(1-\gamma)$. (ii) The score-driven parameter of association ρ_t is determined by using the following transformations of $\tilde{\rho}_t$: For the Clayton and rotated Clayton copulas: $\rho_t = \exp(\tilde{\rho}_t) - 1 \in (-1, \infty)$. For the Gumbel and rotated Gumbel copulas: $\rho_t = \exp(\tilde{\rho}_t) + 1 \in (1, \infty)$. For the Student's t copula: $\rho_t = [1 - \exp(-\tilde{\rho}_t)]/[1 + \exp(-\tilde{\rho}_t)] \in (-1, 1)$.

We estimate all score-driven models in one step, by using the ML method (Harvey 2013; Blasques et al. 2021). In the literature, several works implement two-step estimation procedures for models with copulas (e.g. Bernardi and Catania 2018), in which the parameters of the marginal distributions are estimated in a first step, and the parameters of the copula are estimated in a second step. In the present paper we use a one-step estimation procedure, motivated by the work of Joe (2015).

4 Data

We use daily opening price data for VOO and Bitcoin for the full sample period of September 18, 2014 to January 21, 2022 (source of data: Yahoo Finance). The VOO and Bitcoin daily opening prices are denoted by $p_{1,t}$ and $p_{2,t}$, respectively, which we use to compute daily returns. From the daily returns we subtract the daily expense ratio fees, which in annual terms are 0.03% and 0.95% for VOO and BITO, respectively. The resulting dependent variables are denoted $r_{k,t}$ for $k = 1, 2$. We assume that the risk-free rate is zero. In Table 1, the descriptive statistics of $r_{1,t}$ and $r_{2,t}$ for the full sample period are presented. In Figure 1, the evolution of $p_{1,t}$, $p_{2,t}$, $r_{1,t}$, and $r_{2,t}$ for the full sample period is presented.

We define two investment periods: full investment period (March 21, 2018 to January 21, 2022) and COVID-19 investment period (February 24, 2020 to January 21, 2022). For both investment periods one-step ahead out-of-sample forecasts of expected excess return, volatility, and correlation coefficients are estimated, by using rolling data windows (each with 882 observations). In Table 1, the descriptive statistics for $r_{k,t}$ for the full and COVID-19 investment periods are presented.

For several cases the partial autocorrelation functions (PACFs) are significant, and the ARCH test statistics (Engle 1982) are always significant (Table 1). The first-order autocorrelation $\mathrm{corr}(r_{k,t}, r_{k,t-1})$ estimates are negative in almost all cases, indicating overreaction effects. In the econometric specifications of this paper, we use first-order AR and QAR dynamics for expected excess returns, to control for serial correlation in the mean, and we use first-order dynamics for the volatility filters, to control for ARCH effects. We note that the standard deviation estimates for VOO and Bitcoin are the highest in the COVID-19 investment period (Table 1), and the unconditional volatility estimate of Bitcoin is about four times higher than that of VOO for all sample periods. The impact of $r_{k,t-1}$ on the absolute return of the following trading day $|r_{k,t}|$ (i.e. a proxy of conditional volatility) is always negative (Table 1), which supports the use of leverage effects in all volatility filters of this paper.

[APPROXIMATE LOCATION OF TABLE 1 AND FIGURE 1]

5 Empirical results

In Figure 2, we present the value of 1 USD investments over the full investment period (Figure 2(a)) and the COVID-19 investment period (Figure 2(b)), providing an illustration of the portfolio performances. Those figures focus on the evolution of the VOO, Bitcoin, the 10 best-performing score-driven copula-based strategies, and the best-performing Gaussian-DCC and t-DCC model-based strategies. For both investment periods of Figure 2, the performances of the 10 best-performing score-driven models (black lines) are superior to those of VOO (dark blue lines), Bitcoin (light blue lines), the best-performing Gaussian-DCC model (green lines), and the best-performing t-DCC model (red lines). The model specifications and the portfolio strategies of Figure 2 are presented in Table 2. Some interesting conclusions are obtained from Table 2 on the best-performing score-driven portfolios:

First, for both VOO and Bitcoin, for all but one of the models the marginal distributions are score-driven models (Table 2). This result indicates for equity and BITO portfolio investors that the score-driven marginals are superior to the classical AR-GARCH alternative. Second, for the performances of the score-driven copulas, the portfolio performance results for all investment periods show that the best-performing score-driven copulas are the Gumbel and rotated Gumbel copulas (Table 2). This result indicates for equity and BITO portfolio investors that the Gumbel and rotated Gumbel copulas are superior to the DCC and score-driven Student's t, Clayton, and rotated Clayton copulas. Third, for the portfolio weight updating frequencies, for both investment periods the best-performing portfolios use the semi-annual updating frequency (Table 2). This result indicates for equity and BITO portfolio investors that the semi-annual updating frequency is superior to the weekly, monthly, quarterly, or annual updating frequencies. Fourth, for the portfolio optimization strategies for both investment periods, the best-performing portfolios use either the Sharpe ratio or the utility function (Table 2). This suggests that the minimum-variance portfolio strategy is the worst-performing strategy.

In Table 3, we report statistical test results on the differences between the returns of alternative portfolio strategies. We compare the performances of VOO, Bitcoin, the score-driven portfolios, and the DCC-based portfolios. For each pair of portfolio strategies, the comparison is done by regressing the portfolio return difference on a constant, and by using the ordinary least squares (OLS) estimator with heteroskedasticity and autocorrelation consistent (HAC) standard errors (Newey and West 1987).

For both investment periods, the results in Table 3 show that the performances of several score-driven portfolios, Gaussian-DCC portfolios, and t-DCC portfolios are superior to the performance of VOO. The results also show that VOO is practically never superior to score-driven portfolios, Gaussian-DCC portfolios, and t-DCC portfolios. Finally, Table 3 also shows that the performances of several score-driven portfolios are superior to those of the Gaussian-DCC and t-DCC portfolios.

In Figures 3 and 4, we provide a graphical analysis by classifying the final values of the VOO-Bitcoin portfolios, with respect to the econometric model of marginal distributions of VOO and Bitcoin, the econometric model of association, the objective function of the portfolio optimization problem, and the updating frequency of portfolio weights. In Figures 3 and 4, we present the final portfolio values for the full investment period and the COVID-19 investment period, respectively.

Figures 3 and 4 provide the following conclusions, which may help the investors with the selection of portfolio strategies for VOO and Bitcoin: First, the results of all panels of Figures 3 and 4 are similar. Second, with respect to the models of marginal distributions, the portfolios for score-driven copulas are clearly superior to VOO, Bitcoin, normal-DCC portfolios, and t-DCC portfolios (Figures 3A, 3B, 4A, and 4B). For the marginal distributions of the models with score-driven copulas, AR-t-GARCH and QAR-Beta-t-EGARCH are superior to QAR-Beta-Gen-t-EGARCH (Figures 3A, 3B, 4A, and 4B). Third, with respect to the models of association, the portfolios for score-driven copulas clearly are superior to VOO, Bitcoin, normal-DCC portfolios, and t-DCC portfolios (Figures 3C and 4C). For the score-driven copulas graphically we do not see significant differences among the Clayton, rotated Clayton, Gumbel, rotated Gumbel, and Student's t copulas. Fourth, with respect to the objective function of the portfolio maximization problem, the utility functions with risk aversion coefficients 1 and 4 are similar and superior to the Sharpe ratio, which is highly superior to the minimum-variance portfolio (Figures 3D and 4D). Fifth, with respect to the updating frequencies of portfolio weights, the best strategy is the semi-annual update, followed by the similar annual and quarterly updates, which are followed by the similar monthly and weekly updates (Figures 3E and 4E).

[APPROXIMATE LOCATION OF TABLES 2, 3, AND FIGURES 2, 3, 4]

6 Conclusions

The advantages of the score-driven models may imply better-performing portfolios using score-driven models than classical observation-driven time series models, for the estimation of expected return and volatility of portfolios. The main contribution of this paper is that for the optimization of VOO plus Bitcoin portfolios we use score-driven models, which has never been considered in the literature. Motivated by the recent start of trading of BITO, we have studied whether equity portfolios can be improved by using the BITO cryptocurrency ETF. This is a relevant question because the trading fee of BITO is significantly lower than that the trading fee of Bitcoin, and BITO is traded on a regulated market while the cryptocurrency exchanges on which Bitcoin is traded are largely unregulated. For the equity component of the portfolio, we have used an ETF that replicates the S&P 500 index.

In the empirical application, the full sample period is from September 18, 2014 to January 21, 2022. We have studied the performances of VOO-Bitcoin portfolios for the full investment period of March 21, 2018 to January 21, 2022, and the COVID-19 investment period of February 24, 2020 to January 21, 2022. Portfolio expected return and portfolio volatility forecasting has been performed by using a rolling-window estimation and forecasting approach. We have compared the performances of VOO, 40 portfolios for DCC models, and 900 portfolios for score-driven copulas. We have studied whether the performances of score-driven portfolios are better or worse than the performance of VOO. For the score-driven portfolios, we have used five score-driven copulas, four portfolio optimization strategies, five portfolio weight updating frequencies, and nine combinations of (i) AR-t-GARCH, (ii) QAR-Beta-t-EGARCH, (iii) QAR-Beta-Gen-t-EGARCH. In addition, we have also studied whether the performances of DCC-based portfolios is better or worse than the performance of VOO. For the DCC-based portfolios we have used AR-Gaussian-GARCH and AR-t-GARCH marginals.

We have found that, for both the full and COVID investment periods, the VOO performance is significantly improved by the score-driven portfolios which are also superior to the DCC portfolios.

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Appendix A

In this appendix, for each distribution technical details are presented for the: (i) log conditional density of $r_{k,t}$, (ii) scaled score function $s_{\mu,k,t}$, scale factor $K(\lambda_{k,t})$ for $s_{\mu,k,t}$, and score function $s_{\lambda,k,t}$, (iii) conditional mean and conditional standard deviation of $r_{k,t}$. For the conditioning set, we use $\mathcal{F}_{t-1} = \sigma(r_{k,1}, \ldots, r_{k,t-1} : k = 1, 2)$, and the vector of the constant parameters is denoted Θ.

Student's t-distribution—$\epsilon_{k,t} \sim t[0, 1, \exp(\nu_k) + 2]$ i.i.d., where $\nu_k \in \mathbb{R}$ is a shape parameter, and $\exp(\cdot)$ is the exponential function. (i) The log conditional density of $r_{k,t}$ is

$$\ln f(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \ln\Gamma\left[\frac{\exp(\nu_k)+3}{2}\right] - \ln\Gamma\left[\frac{\exp(\nu_k)+2}{2}\right] \tag{A.1}$$

$$-\frac{\ln(\pi)+\ln[\exp(\nu_k)+2]}{2} - \lambda_{k,t} - \frac{\exp(\nu_k)+3}{2}\ln\left\{1+\frac{\epsilon_{k,t}^2}{\exp(\nu_k)+2}\right\}$$

where $\ln(\cdot)$ is the natural logarithm function and $\Gamma(\cdot)$ is the gamma function. (ii) The score function with respect to $\mu_{k,t}$ is (Harvey 2013):

$$\frac{\partial \ln f(r_{k,t}|\mathcal{F}_{t-1};\Theta)}{\partial \mu_{k,t}} = \frac{[\exp(\nu_k)+2]\exp(\lambda_{k,t})\epsilon_{k,t}}{\epsilon_{k,t}^2+\exp(\nu_k)+2} \times \frac{\exp(\nu_k)+3}{[\exp(\nu_k)+2]\exp(2\lambda_{k,t})} = \tag{A.2}$$

$$= s_{\mu,k,t} \times \frac{\exp(\nu_k)+3}{[\exp(\nu_k)+2]\exp(2\lambda_{k,t})} = s_{\mu,k,t} \times K(\lambda_{k,t})$$

where the scaled score function $s_{\mu,k,t}$ is defined in the second equality, and the scale factor $K(\lambda_{k,t})$ is defined in the last equality. The $s_{\mu,k,t}$ term trims outliers, because $s_{\mu,k,t} \to_p 0$ when $|\epsilon_{k,t}| \to \infty$ (Figure

A1(a)). The score function with respect to $\lambda_{k,t}$ is given by (Harvey and Chakravarty 2008):

$$s_{\lambda,k,t} = \frac{\partial \ln f(r_{k,t}|\mathcal{F}_{t-1};\Theta)}{\partial \lambda_{k,t}} = \frac{[\exp(\nu_k)+3]\epsilon_{k,t}^2}{\exp(\nu_k)+2+\epsilon_{k,t}^2} - 1 \tag{A.3}$$

The updating term $s_{\lambda,k,t}$ Winsorizes extreme observations, because $s_{\lambda,k,t} \rightarrow_p c > 0$ when $|\epsilon_{k,t}| \rightarrow \infty$ (Figure A1(b)). (iii) The conditional mean and standard deviation of $r_{k,t}$, respectively, are

$$E(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \mu_{k,t} \tag{A.4}$$

$$\sigma(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \sigma_{k,t} = \exp(\lambda_{k,t})\left[\frac{\exp(\nu_k)+2}{\exp(\nu_k)}\right]^{1/2} \tag{A.5}$$

Gen-t distribution—$\epsilon_t \sim$ Gen-$t[0,1,\exp(\nu_k)+2,\exp(\eta_k)]$ i.i.d., where $\nu_k \in \mathbb{R}$, and $\eta_k \in \mathbb{R}$ are shape parameters. For $\exp(\eta_k)=2$, the Gen-t distribution is the Student's t-distribution.

(i) The log-density of $r_{k,t}$ is (Ayala, Blazsek, and Escribano 2019):

$$\ln f(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \eta_k - \lambda_{k,t} - \ln(2) - \frac{\ln[\exp(\nu_k)+2]}{\exp(\eta_k)} - \ln\Gamma\left[\frac{\exp(\nu_k)+2}{\exp(\eta_k)}\right] \tag{A.6}$$

$$-\ln\Gamma[\exp(-\eta_k)] + \ln\Gamma\left[\frac{\exp(\nu_k)+3}{\exp(\eta_k)}\right] - \frac{\exp(\nu_k)+3}{\exp(\eta_k)}\ln\left\{1+\frac{|\epsilon_{k,t}|^{\exp(\eta_k)}}{[\exp(\nu_k)+2]}\right\}$$

where sgn(·) is the signum function. (ii) The score function with respect to μ_t is given by:

$$\frac{\partial \ln f(r_{k,t}|\mathcal{F}_{t-1};\Theta)}{\partial \mu_t} = \tag{A.7}$$

$$= \frac{[\exp(\nu_k)+2]\exp(\lambda_{k,t})\epsilon_{k,t}|\epsilon_{k,t}|^{\exp(\eta_k)-2}}{|\epsilon_{k,t}|^{\exp(\eta_k)}+[\exp(\nu_k)+2]} \times \frac{\exp(\nu_k)+3}{[\exp(\nu_k)+2]\exp(2\lambda_{k,t})} =$$

$$= s_{\mu,k,t} \times \frac{\exp(\nu_k)+3}{[\exp(\nu_k)+2]\exp(2\lambda_{k,t})} \equiv u_{\mu,k,t} \times K(\lambda_{k,t})$$

where the scaled score function $s_{\mu,k,t}$ is defined in the second equality, and the scale factor $K(\lambda_{k,t})$ is defined in the last equality. The $s_{\mu,k,t}$ term trims outliers, because $s_{\mu,k,t} \rightarrow_p 0$ when $|\epsilon_{k,t}| \rightarrow \infty$ (Figure A1(c)). The score function with respect to $\lambda_{k,t}$ is (Ayala, Blazsek, and Escribano 2019):

$$s_{\lambda,k,t} = \frac{\partial \ln f(r_{k,t}|\mathcal{F}_{t-1};\Theta)}{\partial \lambda_{k,t}} = \frac{|\epsilon_{k,t}|^{\exp(\eta_k)}[\exp(\nu_k)+3]}{|\epsilon_{k,t}|^{\exp(\eta_k)}+[\exp(\nu_k)+2]} - 1 \tag{A.8}$$

The updating term $s_{\lambda,k,t}$ Winsorizes outliers, because $s_{\lambda,k,t} \to_p c > 0$ when $|\epsilon_{k,t}| \to \infty$ (Figure A1(d)).
(iii) The conditional mean and standard deviation of $r_{k,t}$, respectively, are

$$E(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \mu_{k,t} \tag{A.9}$$

$$\sigma(r_{k,t}|\mathcal{F}_{t-1};\Theta) = \sigma_{k,t} = \exp(\lambda_{k,t})[\exp(\nu_k)+2]^{\exp(-\eta_k)} \times \left\{ \frac{B\left[\frac{3}{\exp(\eta_k)}, \frac{\exp(\nu_k)}{\exp(\eta_k)}\right]}{B\left[\frac{1}{\exp(\eta_k)}, \frac{\exp(\nu_k)+2}{\exp(\eta_k)}\right]} \right\}^{1/2} \tag{A.10}$$

where $B(\cdot,\cdot)$ is the beta function (Ayala, Blazsek, and Escribano 2019).

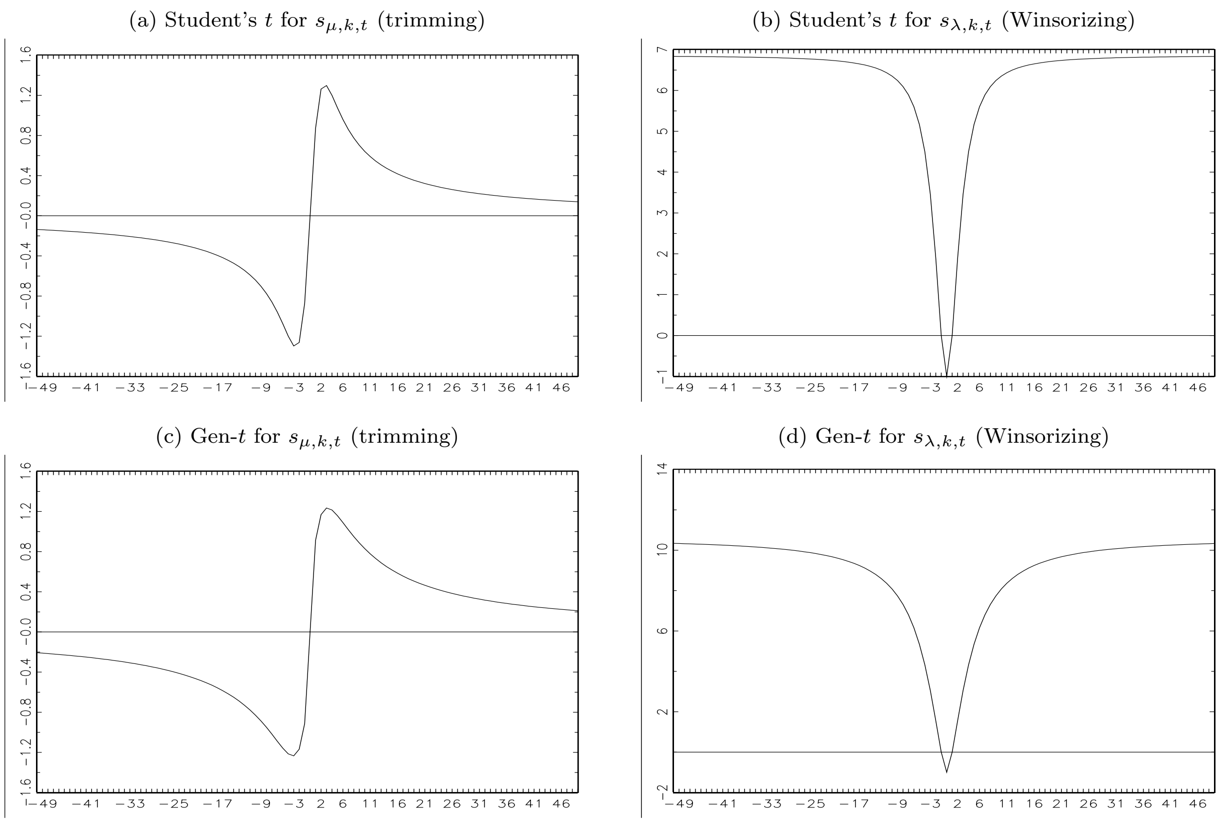

Figure A1. Scaled score function $s_{\mu,k,t}$ and score function $s_{\lambda,k,t}$ estimates, as functions of ϵ_t. *Notes:* ML estimates of the shape parameters with $\lambda_t = 0$ are used. In parentheses, we refer to the asymptotic transformation of outliers, as $|\epsilon_t| \to \infty$.

Appendix B

Clayton copula—The bivariate Clayton copula density function is

$$c_t(u,v) \equiv c(u,v;\rho_t|\mathcal{F}_{t-1};\Theta) = (1+\rho_t)(uv)^{-(\rho_t+1)}(u^{-\rho_t}+v^{-\rho_t}-1)^{-(1+2\rho_t)/\rho_t} \tag{B.1}$$

where u and v are realizations of $U[0,1]$ random variables (we use the same notation for the remaining copulas), and $\rho_t \in [-1,\infty)\backslash\{0\}$ (Harvey 2013; Joe 2015). The partial derivative of $\ln c(u,v;\rho_t)$ is

$$s_{\rho,t} \equiv s_\rho(u,v;\rho_t|\mathcal{F}_{t-1};\Theta) = \frac{\partial \ln c(u,v;\rho_t)}{\partial \rho_t} = \frac{1}{1+\rho_t} - \ln(uv) + \frac{1}{\rho_t^2}\ln\left(u^{-\rho_t}+v^{-\rho_t}-1\right) \tag{B.2}$$

$$+\frac{(1+2\rho_t)[u^{-\rho_t}\ln(u)+v^{-\rho_t}\ln(v)]}{\rho_t(u^{-\rho_t}+v^{-\rho_t}-1)}$$

Rotated Clayton copula—The bivariate rotated Clayton copula density function is

$$c_t(u,v) \equiv c(u,v;\rho_t|\mathcal{F}_{t-1};\Theta) \tag{B.3}$$

$$= (1+\rho_t)[(1-u)(1-v)]^{-(\rho_t+1)}[(1-u)^{-\rho_t}+(1-v)^{-\rho_t}-1]^{-(1+2\rho_t)/\rho_t}$$

with $\rho_t \in [-1,\infty)\backslash\{0\}$ (Patton 2004). The partial derivative of $\ln c(u,v;\rho_t)$ is

$$s_{\rho,t} \equiv s_\rho(u,v;\rho_t|\mathcal{F}_{t-1};\Theta) = \frac{\partial \ln c(u,v;\rho_t)}{\partial \rho_t} = \frac{1}{1+\rho_t} - \ln[(1-u)(1-v)] \tag{B.4}$$

$$+\frac{1}{\rho_t^2}\ln\left[(1-u)^{-\rho_t}+(1-v)^{-\rho_t}-1\right]+$$

$$+\frac{(1+2\rho_t)[(1-u)^{-\rho_t}\ln(1-u)+(1-v)^{-\rho_t}\ln(1-v)]}{\rho_t[(1-u)^{-\rho_t}+(1-v)^{-\rho_t}-1]}$$

Gumbel copula—The bivariate Gumbel copula density function is

$$c_t(u,v) \equiv c(u,v;\rho_t|\mathcal{F}_{t-1};\Theta) = \exp\left\{-\left[(-\ln u)^{\rho_t}+(-\ln v)^{\rho_t}\right]^{1/\rho_t}\right\} \times \tag{B.5}$$

$$\times\frac{[\ln(u)\ln(v)]^{\rho_t-1}}{uv\left[(-\ln u)^{\rho_t}+(-\ln v)^{\rho_t}\right]^{2-1/\rho_t}} \times \left\{\left[(-\ln u)^{\rho_t}+(-\ln v)^{\rho_t}\right]^{1/\rho_t}+\rho_t-1\right\}$$

with $\rho_t \in [1, \infty)$ (Joe 2015; Patton 2004). The partial derivative of $\ln c(u, v; \rho_t)$ is

$$s_{\rho,t} \equiv s_\rho(u, v; \rho_t|\mathcal{F}_{t-1}; \Theta) \tag{B.6}$$

$$= \frac{\partial \ln c(u, v; \rho_t)}{\partial \rho_t} = \frac{\xi_1^{1/\rho_t - 1}}{\rho_t^2} [\xi_1 \ln(\xi_1) - \rho_t \xi_2] + \ln(\ln u \ln v) + \frac{(1 - 2\rho_t)\xi_2}{\rho_t \xi_1}$$

$$-\frac{\ln(\xi_1)}{\rho_t^2} + \frac{\xi_1^{1/\rho_t - 1} [-\xi_1 \ln(\xi_1) + \rho_t \xi_2] + 1}{\rho_t^2 \left(\xi_1^{1/\rho_t} + \rho_t - 1\right)}$$

where $\xi_1 = (-\ln u)^{\rho_t} + (-\ln v)^{\rho_t}$ and $\xi_2 = (-\ln u)^{\rho_t} \ln(-\ln u) + (-\ln v)^{\rho_t} \ln(-\ln v)$.

Rotated Gumbel copula—The bivariate rotated Gumbel copula density function is

$$c_t(u, v) \equiv c(u, v; \rho_t|\mathcal{F}_{t-1}; \Theta) = \exp\left\{- \{[-\ln(1-u)]^{\rho_t} + [-\ln(1-v)]^{\rho_t}]^{1/\rho_t}\right\} \times \tag{B.7}$$

$$\times \frac{[\ln(1-u)\ln(1-v)]^{\rho_t - 1}}{(1-u)(1-v)\{[-\ln(1-u)]^{\rho_t} + [-\ln(1-v)]^{\rho_t}\}^{2 - 1/\rho_t}} \times$$

$$\times \left\{\{[-\ln(1-u)]^{\rho_t} + [-\ln(1-v)]^{\rho_t}]^{1/\rho_t} + \rho_t - 1\right\}$$

with $\rho_t \in [1, \infty)$ (Patton 2004). The partial derivative of $\ln c(u, v; \rho_t)$ is

$$s_{\rho,t} \equiv s_\rho(u, v; \rho_t|\mathcal{F}_{t-1}; \Theta) \tag{B.8}$$

$$= \frac{\partial \ln c(u, v; \rho_t)}{\partial \rho_t} = \frac{\xi_1^{1/\rho_t - 1}}{\rho_t^2} [\xi_1 \ln(\xi_1) - \rho_t \xi_2] + \ln[\ln(1-u)\ln(1-v)]$$

$$+\frac{(1 - 2\rho_t)\xi_2}{\rho_t \xi_1} - \frac{\ln(\xi_1)}{\rho_t^2} + \frac{\xi_1^{1/\rho_t - 1} [-\xi_1 \ln(\xi_1) + \rho_t \xi_2] + 1}{\rho_t^2 \left(\xi_1^{1/\rho_t} + \rho_t - 1\right)}$$

where $\xi_1 = [-\ln(1-u)]^{\rho_t} + [-\ln(1-v)]^{\rho_t}$ and

$$\xi_2 = [-\ln(1-u)]^{\rho_t} \ln[-\ln(1-u)] + [-\ln(1-v)]^{\rho_t} \ln[-\ln(1-v)] \tag{B.9}$$

Student's t copula—The bivariate Student's t-copula density function is

$$c_t(u, v) \equiv c(u, v; \nu, \rho_t|\mathcal{F}_{t-1}; \Theta) = \frac{1}{\sqrt{1 - \rho_t^2}} \frac{\Gamma[(\nu + 2)/2]\Gamma(\nu/2)}{\Gamma^2[(\nu + 1)/2]} \times \tag{B.10}$$

$$\times \frac{\left\{1+\frac{\left[T_\nu^{-1}(u)\right]^2+\left[T_\nu^{-1}(v)\right]^2-2\rho_t T_\nu^{-1}(u)T_\nu^{-1}(v)}{\nu(1-\rho_t^2)}\right\}^{-\frac{\nu+2}{2}}}{\left\{1+\frac{\left[T_\nu^{-1}(u)\right]^2}{\nu}\right\}^{-\frac{\nu+1}{2}}\left\{1+\frac{\left[T_\nu^{-1}(v)\right]^2}{\nu}\right\}^{-\frac{\nu+1}{2}}}$$

where $T_\nu^{-1}(x)$ is the inverse of the Student's t-distribution function, ρ_t is the correlation coefficient, and ν denotes degrees of freedom (Joe 2015). The partial derivative of $\ln c(u, v; \nu, \rho_t)$ is

$$s_{\rho,t} \equiv s_\rho(u, v; \nu, \rho_t|\mathcal{F}_{t-1}; \Theta) = \frac{\partial \ln c(u, v; \nu, \rho_t)}{\partial \rho_t} = \frac{\rho_t}{1-\rho_t^2} + \tag{B.11}$$

$$+\frac{\nu+2}{\rho_t^2-1} \times \frac{\rho_t\left\{\left[T_\nu^{-1}(u)\right]^2+\left[T_\nu^{-1}(v)\right]^2\right\}-(\rho_t^2+1)T_\nu^{-1}(u)T_\nu^{-1}(v)}{\left[T_\nu^{-1}(u)\right]^2+\left[T_\nu^{-1}(v)\right]^2-2\rho_t T_\nu^{-1}(u)T_\nu^{-1}(v)-\nu(\rho_t^2-1)}$$

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Table 1: Descriptive statistics.

A.	Full sample		Full investment period		COVID-19 investment period	
Start date	September 18, 2014		March 21, 2018		February 24, 2020	
End date	January 21, 2022		January 21, 2022		January 21, 2022	
Sample size T	1850		968		484	
B.	**VOO**	**Bitcoin**	**VOO**	**Bitcoin**	**VOO**	**Bitcoin**
Minimum	-0.0846	-0.3659	-0.0846	-0.3659	-0.0846	-0.3659
Maximum	0.0631	0.2769	0.0631	0.2769	0.0631	0.1876
Mean	0.0005	0.0035	0.0006	0.0026	0.0007	0.0042
Standard deviation	0.0103	0.0459	0.0118	0.0449	0.0142	0.0480
Skewness	-0.9591	0.0200	-0.9353	-0.1813	-0.9824	-0.8169
Excess kurtosis	9.4909	6.0637	7.9943	7.2829	6.7408	8.1158
PACF(1)	-0.0669^{***}	0.0100	-0.0659^{**}	-0.0322	-0.0733	-0.0996^{**}
PACF(2)	0.0127	0.0129	0.0377	0.0699^{**}	0.0390	0.0938^{**}
PACF(3)	0.0549^{**}	0.0169	0.0411	0.0241	0.0399	0.0211
ARCH(5) statistic	344.3170^{***}	64.9012^{***}	220.3270^{***}	16.7539^{***}	110.4200^{***}	15.2066^{***}
$\mathrm{corr}(r_{k,t}, r_{k,t-1})$	-0.0670	0.0100	-0.0660	-0.0322	-0.0739	-0.0996
$\mathrm{corr}(\lvert r_{k,t}\rvert, r_{k,t-1})$	-0.1206	-0.0367	-0.0801	-0.0185	-0.0709	-0.0531

Partial autocorrelation function (PACF); autoregressive conditional heteroskedasticity (ARCH); coronavirus pandemic of 2019 (COVID-19). Daily opening prices are denoted $p_{k,t}$ for $k \in \{1,2\} \equiv \{\text{VOO}, \text{Bitcoin}\}$. Daily returns are denoted $r_{k,t} = (p_{k,t} - p_{k,t-1})/p_{k,t-1}$ for $k \in \{1,2\}$. We assume that the risk-free rate is zero. From the daily excess returns, we subtract the fees corresponding to the 0.03% annual expense ratio of VOO and the 0.95% annual expense ratio of BITO, respectively. The PACF and ARCH lag-orders are reported in parentheses. **, and *** show significance at the 5%, and 1% levels, respectively.

Table 2: Ranking of score-driven portfolio performances by using portfolio values at the end of the investment periods.

A. Full investment period (March 21, 2018 to January 21, 2022)						
Ranking	**VOO marginal**	**Bitcoin marginal**	**Copula**	**Update**	**Portfolio strategy**	**Value**
1	QAR-Beta-t-EGARCH	AR-t-GARCH	Gumbel	Semi-annual	Mean-Var, Sharpe	10.2912
2	QAR-Gen-t-EGARCH	QAR-Beta-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 4	10.1072
3	QAR-Beta-t-EGARCH	QAR-Beta-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 4	10.0601
4	QAR-Beta-t-EGARCH	QAR-Beta-t-EGARCH	Student's t	Semi-annual	Mean-Var, RA = 1	9.9926
5	QAR-Gen-t-EGARCH	QAR-Beta-t-EGARCH	Student's t	Semi-annual	Mean-Var, RA = 1	9.9758
6	QAR-Gen-t-EGARCH	QAR-Beta-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 1	9.9353
7	QAR-Gen-t-EGARCH	QAR-Beta-t-EGARCH	Gumbel	Semi-annual	Mean-Var, RA = 1	9.9341
8	QAR-Beta-t-EGARCH	QAR-Beta-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 1	9.9270
9	QAR-Gen-t-EGARCH	QAR-Beta-t-EGARCH	Clayton	Semi-annual	Mean-Var, RA = 1	9.9108
10	QAR-Gen-t-EGARCH	QAR-Gen-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 4	9.8486
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Bitcoin						4.5538
t-DCC	AR-t-GARCH	AR-t-GARCH	t-DCC	Quarterly	Mean-Var, RA = 1	2.6380
normal-DCC	AR-normal-GARCH	AR-normal-GARCH	normal-DCC	Quarterly	Mean-Var, RA = 1	2.0819
VOO						1.6392
B. COVID-19 investment period (February 24, 2020 to January 21, 2022)						
Ranking	**VOO marginal**	**Bitcoin marginal**	**Copula**	**Update**	**Portfolio strategy**	**Value**
1	QAR-Gen-t-EGARCH	QAR-Beta-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 4	4.8814
2	QAR-Beta-t-EGARCH	QAR-Beta-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 4	4.8607
3	QAR-Gen-t-EGARCH	QAR-Gen-t-EGARCH	Gumbel	Semi-annual	Mean-Var, RA = 1	4.8521
4	QAR-Gen-t-EGARCH	QAR-Gen-t-EGARCH	Gumbel	Semi-annual	Mean-Var, Sharpe	4.8520
5	QAR-Gen-t-EGARCH	QAR-Gen-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, Sharpe	4.8518
6	QAR-Gen-t-EGARCH	QAR-Gen-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 1	4.8508
7	QAR-Beta-t-EGARCH	QAR-Gen-t-EGARCH	R Gumbel	Semi-annual	Mean-Var, RA = 1	4.8459
8	QAR-Gen-t-EGARCH	QAR-Gen-t-EGARCH	Clayton	Semi-annual	Mean-Var, RA = 1	4.8425
9	QAR-Gen-t-EGARCH	QAR-Gen-t-EGARCH	Student's t	Semi-annual	Mean-Var, RA = 1	4.8397
10	QAR-Beta-t-EGARCH	QAR-Gen-t-EGARCH	Gumbel	Semi-annual	Mean-Var, RA = 1	4.8387
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Bitcoin						4.0511
VOO						1.3791
t-DCC	AR-t-GARCH	AR-t-GARCH	t-DCC	Annual	Mean-Var, RA = 1	1.3507
normal-DCC	AR-normal-GARCH	AR-normal-GARCH	normal-DCC	Semi-annual	Mean-Var, RA = 1	1.2770

Quasi-autoregressive (QAR); exponential generalized AR conditional heteroskedasticity (EGARCH); risk aversion (RA); coronavirus pandemic of 2019 (COVID-19); Clayton copula (Clayton); rotated Clayton copula (R Clayton); Gumbel copula (Gumbel); rotated Gumbel copula (R Gumbel); Student's t-copula (Student's t); mean-variance strategy for the Sharpe ratio (Mean-Var, Sharpe); mean-variance strategy for a utility function (Mean-Var, RA). In the table, normal-DCC and t-DCC show the performances of the best-performing normal-DCC and t-DCC portfolios, respectively.

Table 3: Statistical comparison of the performances of 900 score-driven portfolios, 2×20 DCC portfolios, and VOO.

A. Full investment period (March 21, 2018 to January 21, 2022)	**Count**	**%**
Score-driven portfolios are superior to VOO	225 out of 900	25%
normal-DCC portfolios are superior to VOO	6 out of 20	30%
t-DCC portfolios are superior to VOO	7 out of 20	35%
Score-driven portfolios are inferior to VOO	0 out of 900	0%
normal-DCC portfolios are inferior to VOO	0 out of 20	0%
t-DCC portfolios are inferior to VOO	1 out of 20	5%
Score-driven portfolios are superior to the best-performing normal-DCC portfolio	119 out of 900	13%
Score-driven portfolios are superior to the best-performing t-DCC portfolio	49 out of 900	5%
B. Covid investment period (February 24, 2020 to January 21, 2022)	**Count**	**%**
Score-driven portfolios are superior to VOO	433 out of 900	48%
normal-DCC portfolios are superior to VOO	10 out of 20	50%
t-DCC portfolios are superior to VOO	11 out of 20	55%
Score-driven portfolios are inferior to VOO	0 out of 900	0%
normal-DCC portfolios are inferior to VOO	0 out of 20	0%
t-DCC portfolios are inferior to VOO	0 out of 20	0%
Score-driven portfolios are superior to the best-performing normal-DCC portfolio	145 out of 900	16%
Score-driven portfolios are superior to the best-performing t-DCC portfolio	90 out of 900	10%

Dynamic conditional correlation (DCC). We show the number of statistically superior portfolios and their proportions, at the 10% level of significance. For testing the differences between the performances of the alternative portfolios, we use an OLS-HAC (ordinary least squares, heteroskedasticity and autocorrelation consistent) estimator for portfolio return differences.

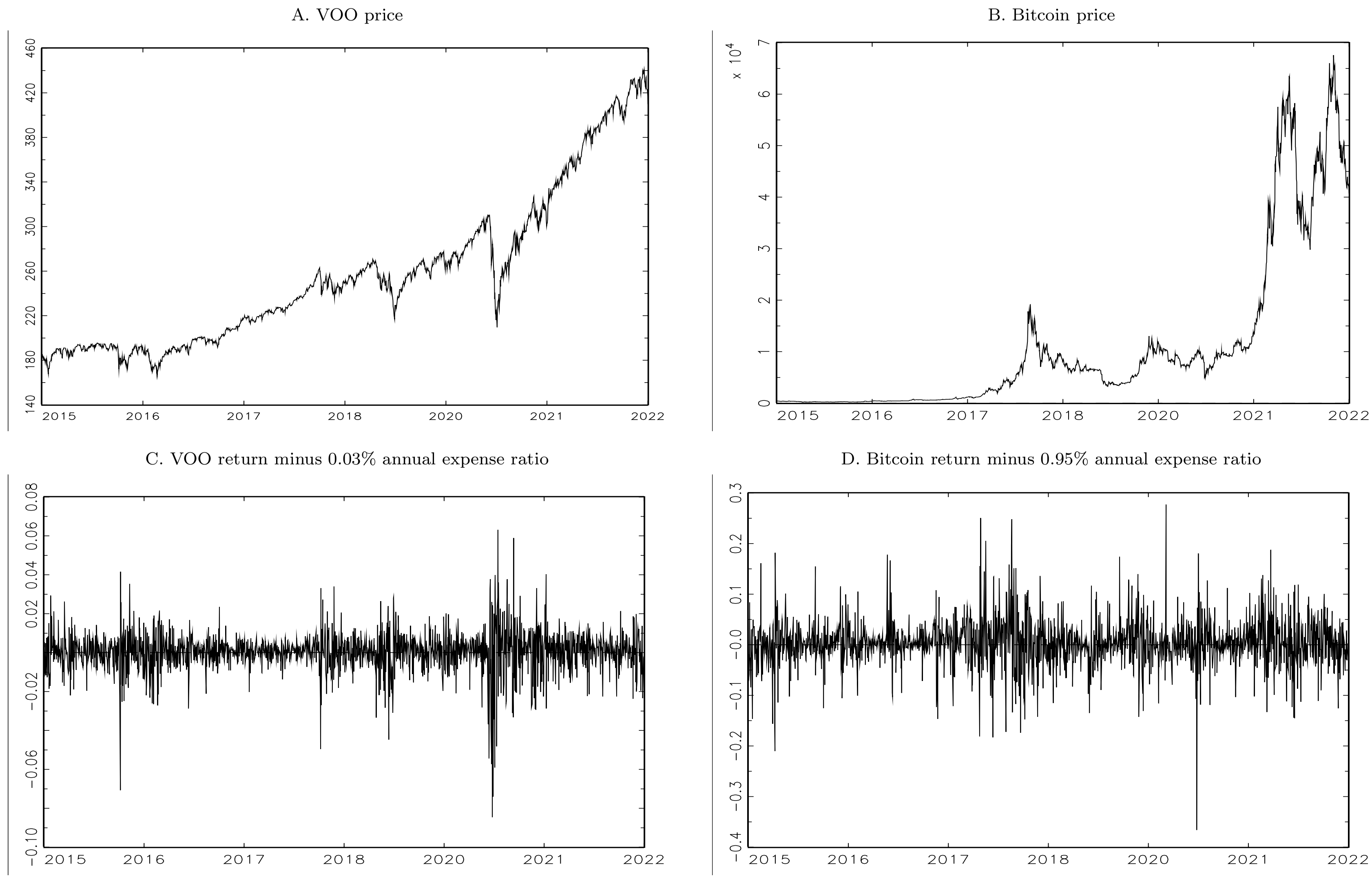
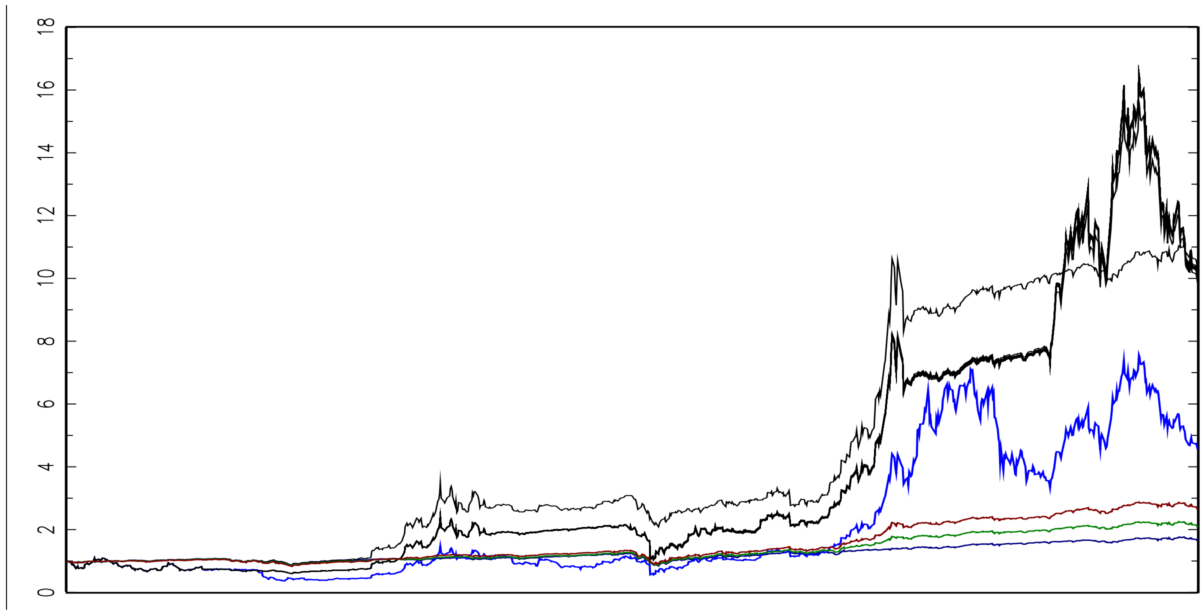


Figure 1: Evolution of VOO and Bitcoin prices and daily returns (from September 18, 2014 to January 21, 2022).

A. Full investment period (March 21, 2018 to January 21, 2022)



B. COVID-19 investment period (February 24, 2020 to January 21, 2022)

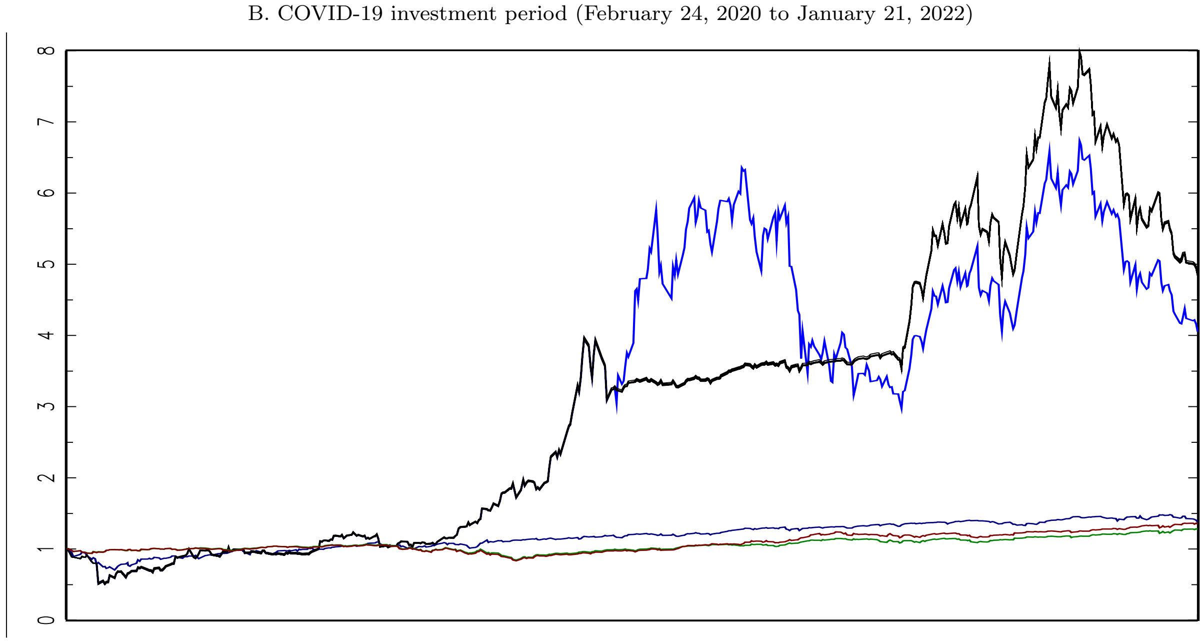


Figure 2: Value of 1 USD investments for VOO (dark blue), Bitcoin (light blue), the best-performing Gaussian-DCC portfolio (green), the best-performing t-DCC portfolio (red), and the 10 best-performing score-driven portfolios (black).

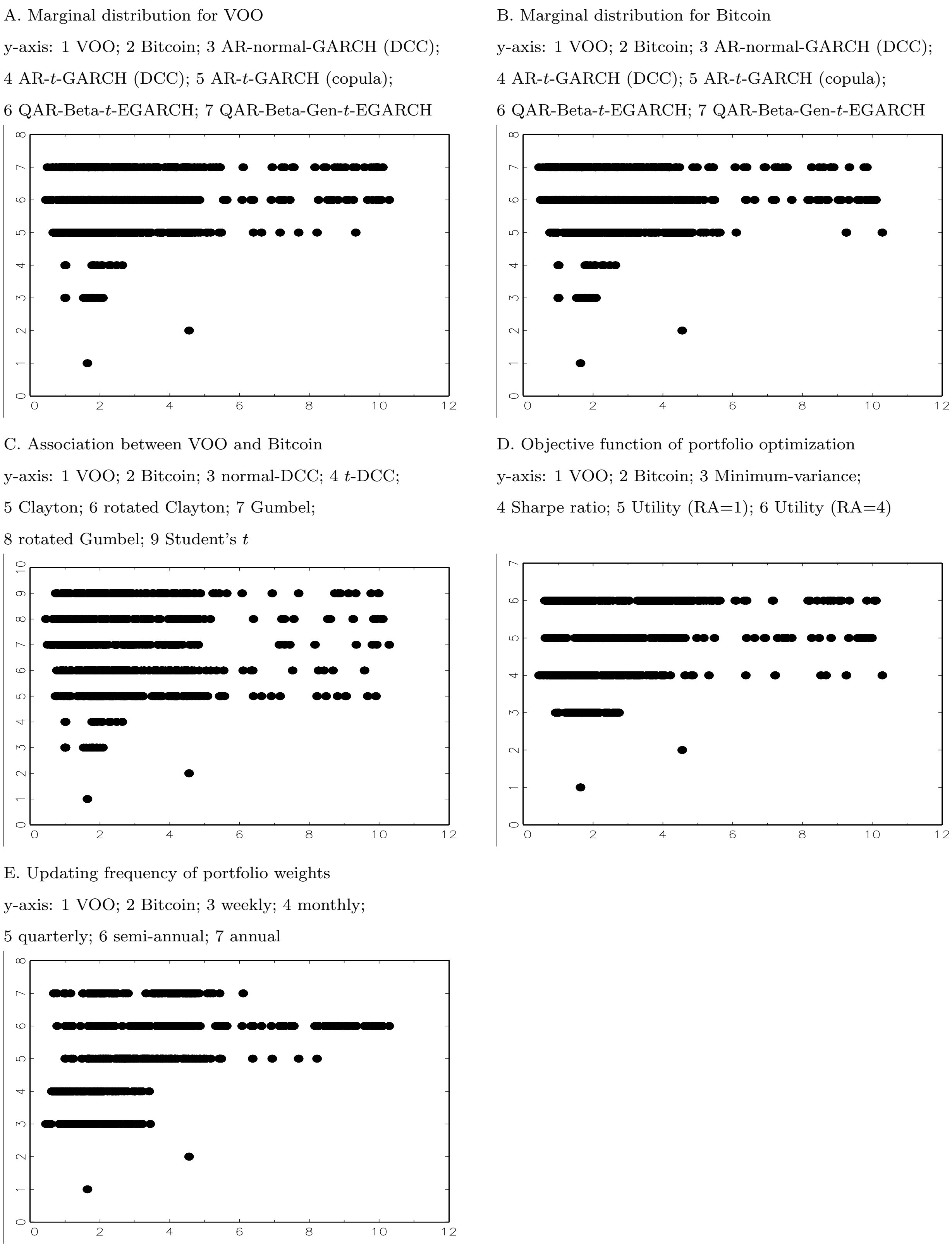
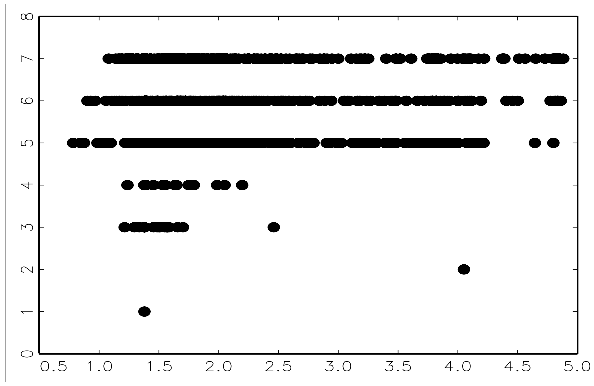


Figure 3: Values of a 1 USD investment on January 21, 2022 (x-axis) for VOO, Bitcoin, 20 normal-DCC portfolios, 20 t-DCC portfolios, and 900 score-driven portfolios, for the full investment period of March 21, 2018 to January 21, 2022.

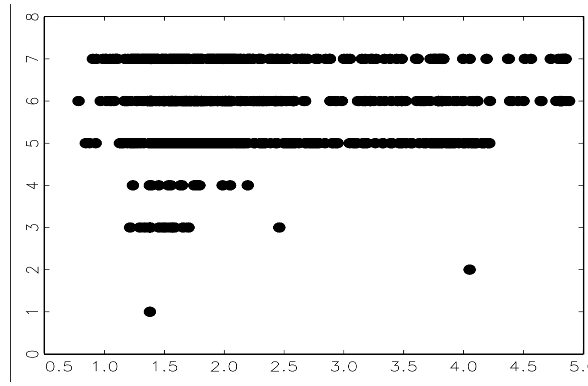
A. Marginal distribution for VOO

y-axis: 1 VOO; 2 Bitcoin; 3 AR-normal-GARCH (DCC); 4 AR-t-GARCH (DCC); 5 AR-t-GARCH (copula); 6 QAR-Beta-t-EGARCH; 7 QAR-Beta-Gen-t-EGARCH



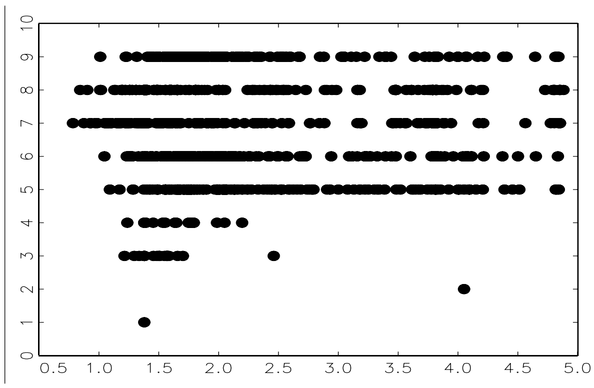
B. Marginal distribution for Bitcoin

y-axis: 1 VOO; 2 Bitcoin; 3 AR-normal-GARCH (DCC); 4 AR-t-GARCH (DCC); 5 AR-t-GARCH (copula); 6 QAR-Beta-t-EGARCH; 7 QAR-Beta-Gen-t-EGARCH



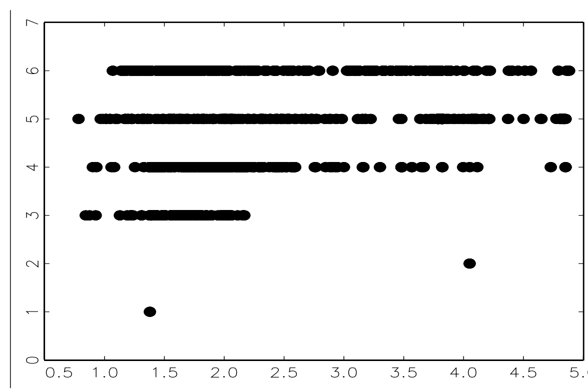
C. Association between VOO and Bitcoin

y-axis: 1 VOO; 2 Bitcoin; 3 normal-DCC; 4 t-DCC; 5 Clayton; 6 rotated Clayton; 7 Gumbel; 8 rotated Gumbel; 9 Student's t



D. Objective function of portfolio optimization

y-axis: 1 VOO; 2 Bitcoin; 3 Minimum-variance; 4 Sharpe ratio; 5 Utility (RA=1); 6 Utility (RA=4)



E. Updating frequency of portfolio weights

y-axis: 1 VOO; 2 Bitcoin; 3 weekly; 4 monthly; 5 quarterly; 6 semi-annual; 7 annual

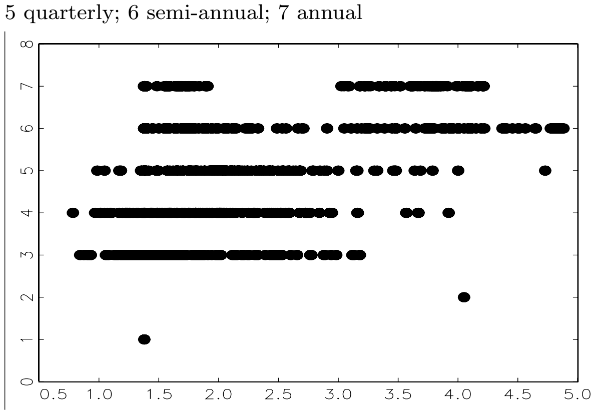


Figure 4: Values of a 1 USD investment on January 21, 2022 (x-axis) for VOO, Bitcoin, 20 normal-DCC portfolios, 20 t-DCC portfolios, and 900 score-driven portfolios, for the COVID-19 investment period of February 24, 2020 to January 21, 2022.