Two-bidder all-pay auctions with interdependent valuations, including the highly competitive case

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Idea

Motivation

- The study of contests – games in which participants irrevocably expend resources in the pursuit of (a) prize(s) – is recently quite an active field.
- Applications: R&D, electoral competition, lobbying, international relations and war, sports ...
- A technical challenge: When players ex ante expect competition to be strong, equilibrium can be difficult to identify or characterise.

Abstract

- Two-bidder all-pay auctions with discrete signals and interdependent valuations.
- We do not require any monotonicity conditions.
- This allows us to analyse "highly competitive" environments.
- Equilibria can be non-monotonic with rich, complex structure.
- We provide an algorithm that can construct all equilibria.

Model

Basic setup

- Two (ex-ante symmetric) bidders competing for a single indivisible object.
- Each bidder receives a signal from a finite set $S = \{s_k\}$.
- If one bidder receives s_k, the probability the other bidder gets signal s_l is $h_{l|k}$.
- The expected value to a bidder of the object, conditional on the bidder getting s_k and the other bidder s_l is $V_{k,l}$.
- Bidders simultaneously submit bids, which they pay irrespective of the outcome.
- The bidder submitting the higher bid wins the object; ties are broken at random.

Baye, Kovenock, de Vries 1996

- Case of commonly-known common-value is a special case of Baye et al (1996).
- Let the common value be V
- Then, equilibrium is uniform randomisation over $[0, V]$.

Siegel 2014

- Siegel (2014) assumes a monotonicity condition (similar to Krishna-Morgan 1997)
- If $h_{l|k}V_{k,l}$ is monotonic in s_l for all s_k, then no two signals are "active" at the same bid.
- In equilibrium, signals sort. In the symmetric case,
  - Each signal randomises over a different interval of bids
  - On each interval, the bidder randomises uniformly as in Baye et al.

Minding the gap

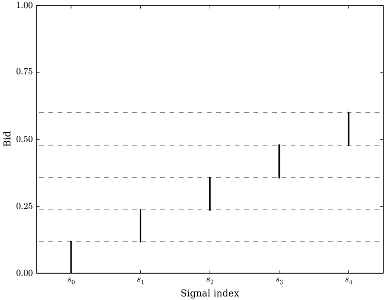
- What does it mean for $h_{l|k} V_{k,l}$ to be monotonic in s_l for all s_k?
- Siegel: this is a condition which says that "signals are not too affiliated."
- But this would rule out, for example, some interesting perturbations of Baye et al setting.
- Our contribution is that we fill this gap.

Example: "Value or noise"

- Consider the case of pure common values, with K possible values $v_k = \frac{k}{K}$.
- Conditional on value v_k, each bidder gets an independent signal, which is s_k with probability p_c. Other signals $s_l \neq s_k$ occur with probability p_w, with $p_c + (K-1)p_w = 1$.
- Monotonicity condition holds iff $\frac{p_w}{p_c} > \frac{v_k}{v_{k+1}}$ for all k, that is, if signals are not too accurate.

"Value or noise": Inaccurate signals ($p_C = 0.21$)

- Pick $K = 5$ and $p_C = 0.21$.
- Then the Krishna-Morgan-Siegel monotonicity condition is satisfied.
- Signals are sorted in equilibrium.

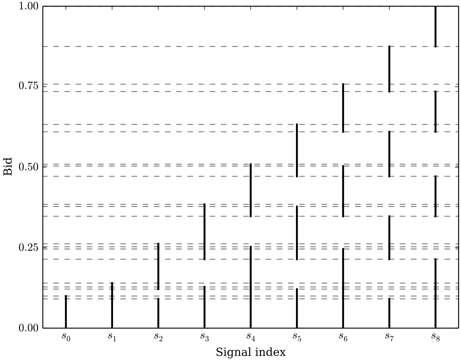


Example: “Value plus noise”

- Pure common values, with K possible values $v_k = \frac{k}{K}$.
- Conditional on value v_k, each bidder gets an independent signal, which can be s_{k-1} or s_k with equal probability.
- A discretised version of models where the signals are on $[v - \delta, v + \delta]$ (e.g. Casari et al 2007 for first-price auctions).
- Monotonicity condition does not hold for any choice of K.

“Value plus noise”: A sneak peek

- This is a the structure of the support of the equilibria in this model, with 8 values and 9 signals:



Equilibrium

Equilibrium

Theorem

A Nash equilibrium exists, in which there are no mass points.

- Existence is not immediate because of the discontinuity in the payoff function at the possibility of a tie.
- Proof idea: Consider perturbed games in which a bid of b is modified by noise $[b-\varepsilon, b+\varepsilon]$.
- Construct a convergent sequence of equilibria as $\varepsilon \downarrow 0$.
- The limit behaviour strategy cannot have mass points (intuition: because if there is too much probability mass in an interval, a bidder will always want to deviate upwards to a higher bid).

Intervals

- Fix some equilibrium behaviour strategy profile π, with π_k being the strategy conditional on signal s_k.
- Let $u_k(b|\pi)$ be the payoff to bidding b against π conditional on signal s_k.
- Because π has no mass points, $u_k(\cdot|\pi)$ is a continuous function.
- Write the set of best response bids can be written as the union of a collection of disjoint closed intervals.

Intervals

- Any equilibrium π therefore divides up the set of possible bids into a collection of intervals $\{\mathcal{I}_j\}$.
- Associated with each interval $\mathcal{I}_j$ is a subset $A_j \subseteq S$ of signals, such that any bid in $\mathcal{I}_j$ is a best response for any signal in A_j.
- We can say things about:
  - Which subsets of S can possibly appear in an equilibrium;
  - The *order* in which those subsets can appear.

Admissible active sets

- If bids in b in some interval $\mathcal{I}$ are all best responses for a signal s_k, it must be that

$$\sum_{s_l \in S} h_{l|k} V_{k,l} \pi_l = 1 \;\; \forall s_k \in A, \; \forall b \in \mathcal{I}.$$

- If this linear system of equations has a solution with $\pi_k \geq 0$ for all, we say that the set of signals A is an **admissible active set**.
- (Technical aside: Solutions with non-constant densities on an interval are possible. But if a solution exists, a constant-density solution exists; focus on those for the moment.)

Admissible active sets: Observations

- The admissibility of an active set is independent of the bid level b.
- Associated with an admissible active set is a set of solutions for the length of the corresponding interval, and the probability mass each signal's strategy assigns to that interval.
- This set is the solution of a set of linear equations - therefore computationally straightforward.

Ordering active sets

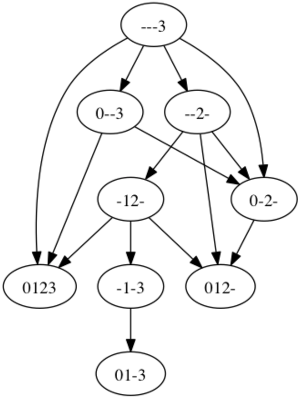
- Active sets cannot appear in arbitrary order in an equilibrium.
- Consider two intervals $\mathcal{I}_j$ and $\mathcal{I}_{j+1}$.
  - Suppose a signal $s_k \in \mathcal{I}_j$ but $s_k \notin \mathcal{I}_{j+1}$. Then, the payoff to s_k on $\mathcal{I}_{j+1}$ must be decreasing.
  - Suppose a signal $s_k \notin \mathcal{I}_{j+1}$ but $s_k \in \mathcal{I}_{j+1}$. Then, the payoff to s_k on $\mathcal{I}_j$ must be increasing.
- These constraints then induce a relation $\succ$ between any two active sets A and A', with $A \succ A'$ if, in an equilibrium A can be active on some interval $\mathcal{I}_j$ and A' on the next (lower) interval $\mathcal{I}_{j+1}$.

Ordering active sets

- The relation $\succ$ induces a graph over admissible active sets.
- Any equilibrium then corresponds to some path through the graph induced by $\succ$.
- The KMS monotonicity condition implies admissible active sets are singletons, and there is a unique path through the graph.
- We show when the KMS monotonicity condition does not hold, then this graph can become quite complex.

Ordering active sets: Example

- Take the "value plus noise" model, where there are $K = 3$ possible values (and therefore 4 signals).



Ordering active sets: Constraints

- A signal must receive the same payoff on all intervals on which it bids.
- If there is a gap in a signal's support, the net payoff change over that interval of bids must be zero.
- This adds an additional linear constraint on the lengths of the intervals involved in such a gap.

The construction

To construct an equilibrium, then:

- Pick a path through the graph induced by $\succ$
- For each active set on the path, construct the linear equations supporting that active set.
- For each signal, for each gap in that signal's support, construct the linear equation ensuring the net payoff on the gap is zero.
- Solve this set of linear equations.
- If the resulting solution has sensible densities, interval lengths, and gives all bidders a non-negative payoff, it's an equilibrium.

Complexity

- The algorithm of Siegel (2014) is a special case of ours.
- Our algorithm can be extended to non-symmetric games; but we will see things are quite complex enough as it is!
- Where Siegel's algorithm applies, the graph of $\succ$ amounts to a single path.
- In our algorithm, a path corresponds to a (rather low-dimensional) system of linear equations; not hard to solve on its own.
- The complexity lies in the massive number of paths through the graph of $\succ$, which in general cannot be pruned efficiently.

Examples

Examples

- I'll now talk about the "value or noise" and "value plus noise" examples in a bit more detail.
- The full paper has a few other examples, including
  - Examples with interdependent (rather than pure common) values;
  - Examples with multiple equilibrium supports, corresponding to different paths through the graph of $\succ$;
  - Examples where the the maximum bid submitted in equilibrium is not monotonic in the posterior expected value of the object.

Example: "Value or noise"

- Pure common values, with K possible values $v_k = \frac{k}{K}$.
- Conditional on value v_k, each bidder gets an independent signal, which is s_k with probability p_c. Other signals $s_l \neq s_k$ occur with probability p_w, with $p_c + (K-1)p_w = 1$.
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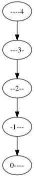
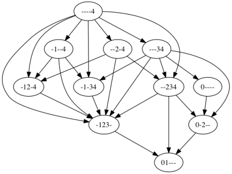
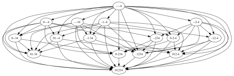
“Value or noise”: Admissible active sets

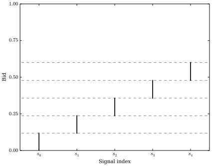




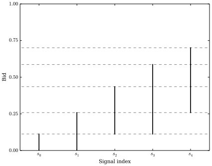


$p_c = 0.21$ $p_c = 0.45$ $p_c = 0.99$

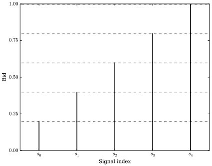
“Value or noise”: Equilibrium support



$p_c = 0.21$



$p_c = 0.45$



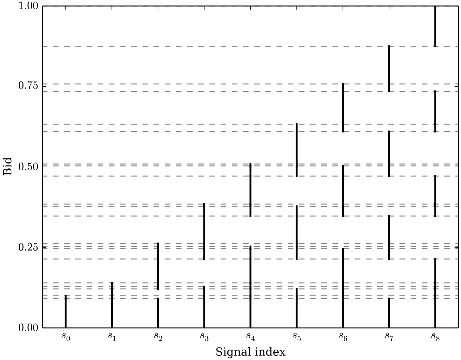
$p_c = 0.99$

"Value or noise": Equilibrium selection

- In the limit as $p_c = p_W$, this game amounts to the all-pay auction with commonly-known values. (Baye et al)
- Equilibrium in Baye et al is uniform randomisation between 0 and the expected value of the object.
- However, the limiting equilibrium in the value-or-noise model is separating.
- Note that the separating equilibrium **is** an equilibrium of the Baye et al model with informationless signals; the informationless signal can still serve as a "coordination" device across different bidder signals.
- This is a counterintuitive selection result. (Counterintuitive to us at least.)

"Value plus noise"

- Recall the schematic diagram of the structure of the equilibrium in this model, with 8 values and 9 signals:



"Value plus noise": Characterisation of equilibrium

- We can show that equilibrium in this game necessarily has the same qualitative structure of diagonal "bands" for any K.
- Note that if some signal s_k is inactive, then the equilibrium conditions for signals s_{k-1} and below are independent of those of signals s_{k+1} and above.
- We therefore can decompose active sets into **isolated groups** of signals which are active, and analyse those isolated groups separately.

"Value plus noise": Characterisation of equilbrium

Theorem

An isolated group in any admissible active set falls into these categories:

1. *Singleton signals;*
2. *Even-parity groups of adjacent interior signals (i.e., excluding s_0 and s_K);*
3. *All signals from s_0 to s_k for any $1 \leq k \leq K$.*

- Fun aside: Proof of (2) is done by Farkas' Lemma.

"Value plus noise": Characterisation of equilibrium

Theorem

The active set consisting of all signals is always admissible, and always admits a multiplicity of solutions. The set of admissible densities supporting the active set converges to a point as $K \to \infty$, with $\pi_0 = \ln 2$.

- That is, equilibrium is not unique.
- However, for examples we have investigated, equilibrium is unique up to the choice of randomisation densities on the interval just above a bid of zero, on which all signals are active.

Summary

Abstract

- Two-bidder all-pay auctions with discrete signals and interdependent valuations.
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- This allows us to analyse “highly competitive” environments.
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