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The predictive performance of AR plus GARCH versus QAR plus Beta-t-EGARCH for extreme observations

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Summary: In this manuscript two dynamic econometric models are compared. The first model is a standard financial time-series model of asset returns: the AR (autoregressive) plus GARCH (generalized autoregressive conditional heteroscedasticity) model. The second model is a recent financial time-series model of asset returns that belongs to the family of dynamic conditional score (DCS) models: the QAR (quasi-AR) plus Beta-t-EGARCH (exponential GARCH) model. A general property of DCS models is that the effects of extreme observations are reduced, hence a DCS model is robust to extreme observations. For DCS models the degree of discounting of extreme observations is endogenously estimated. With respect to the treatment of extreme observations, AR-GARCH is a special case of QAR-Beta-t-EGARCH. In the present paper, we compare the return and volatility predictive performances of AR-GARCH and QAR-Beta-t-EGARCH. The main purpose of this study is to compare those predictive performances for the days when extreme value is observed and also for the first trading day after days when extreme value is observed. The discussion paper is organized in two chapters. In the first chapter, we use data from the Standard & Poor's 500 (S&P 500) index for period 1950 to 2016. We study the predictive performances of AR-GARCH and QAR-Beta-t-EGARCH for all days of the data window, for the days when extreme value is observed and for the first trading day after days when extreme value is observed. In the second chapter, we use an extended time period and an extended econometric specification. We use historical data from the Dow Jones Industrial Average (DJIA) index for period 1896 to 2017. We study the return and volatility predictive

performances of AR-GARCH with leverage effects and QAR-Beta-t-EGARCH with leverage effects for the days when extreme value is observed and for the first trading day after days when extreme value is observed. In both chapters, we define extreme observations by using the Chebyshev inequality. The most important result of this discussion paper is that AR-GARCH dominates QAR-Beta-t-EGARCH for the days when extreme observation is observed, and QAR-Beta-t-EGARCH dominates AR-GARCH for the first trading day after days when extreme value is observed. This result provides a suggestion to financial investors with respect to the choice of the financial time-series model that is applied for return and volatility predictions, for the days when extreme observation is observed and for the consecutive trading day.

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Chapter 1

Should market value news be cause for concern? A study on AR plus GARCH versus QAR plus Beta-t-EGARCH

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Abstract. The manner in which investors react to incoming market news can present a bias in their choice of algorithm as a means of effectively utilising that news. In this paper, we study whether to be concerned or not after news on market value, and use either the autoregressive (AR) plus generalized autoregressive conditional heteroscedasticity (GARCH) or quasi-AR (QAR) plus Beta-t-EGARCH (exponential GARCH) models, respectively. We use data for period 1950 to 2016 from the Standard & Poor's 500 (S&P 500) index. We use the following datasets: (D1) all days of the in-sample data window, (D2) each day for which an outlier is observed, and (D3) the trading day after each day for which an outlier is observed. We use alternative definitions of outliers, according to Chebyshev's inequality. We obtain the following results. For (D1), it is better to be calm and use QAR plus Beta-t-EGARCH. For (D2), it is better to be concerned and use AR plus GARCH. For (D3), it is better to be calm and use QAR plus Beta-t-EGARCH.

Keywords: dynamic conditional score (DCS) models; autoregressive (AR) plus generalized autoregressive conditional heteroscedasticity (GARCH) model; quasi-AR (QAR) plus Beta-t-EGARCH (exponential GARCH) model; outliers; Chebyshev's inequality

JEL classification: C22, C52, C58, G12

I. Introduction

In this paper, we compare the in-sample statistical and in-sample forecast performances of autoregressive (AR) (Box and Jenkins, 1970) plus generalized autoregressive conditional heteroscedasticity (GARCH) (Bollerslev, 1986; Taylor, 1986) and quasi-AR (QAR) (Harvey, 2013) plus Beta-t-EGARCH (exponential GARCH) (Harvey and Chakravarty, 2008) models. AR plus GARCH is a standard financial time-series model, and QAR plus Beta-t-EGARCH belongs to the family of dynamic conditional score (DCS) models (Creal et al., 2013; Harvey, 2013). For DCS models, each dynamic equation is updated by the conditional score of the log-likelihood (LL) function with respect to a time-varying parameter.

The manner in which investors react to incoming market news can affect the model they might adopt in order to effectively deal with that news. An important difference between AR plus GARCH and QAR plus Beta-t-EGARCH models is how expected return and volatility are updated after the new information ϵ_t arrives to the market. For AR plus GARCH, expected return and volatility are updated proportionally to ϵ_t and ϵ_t^2, respectively. Hence, ϵ_t is not discounted in the case of AR due to the linear transformation, and it is accentuated for GARCH due to the quadratic transformation. For QAR plus Beta-t-EGARCH, expected return and volatility are updated proportionally to different non-linear transformations of ϵ_t. Those non-linear transformations discount ϵ_t for both expected return and volatility. The purpose of this paper is to study whether to be concerned or not after news on market value ϵ_t, and use either AR plus GARCH or QAR plus Beta-t-EGARCH, respectively.

We use data from the Standard & Poor's 500 (S&P 500) market index. The in-sample data window is for period 1950 to 2016. In our empirical analysis, we use the following datasets: (D1) all days of the in-sample data window, (D2) each day for which an outlier is observed, and (D3) the trading day after each day for which an outlier is observed. For (D2) and (D3), we use alternative definitions of outliers, according to Chebyshev's inequality.

We obtain the following results. First, for (D1), the LL-based performance of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH. Correspondingly, both return and volatility predictions of QAR plus Beta-t-EGARCH are superior to those of AR plus GARCH. This result suggests that, in general, it is better to be calm and use QAR plus Beta-t-EGARCH, instead of being concerned and using AR plus GARCH.

Second, for (D2), the volatility prediction of AR plus GARCH is superior to that of QAR plus Beta-t-EGARCH, for all outlier definitions. This suggests that volatility can be predicted better by being concerned and using AR plus GARCH for prediction. According to this result, one would need to know a priori that tomorrow there will be an outlier, and given that information one would then use AR plus GARCH.

Third, for (D3), the return prediction of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH, for all outlier definitions. For (D3), we also find that the volatility prediction of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH, with respect to one of the outlier definitions. According to these results, it is better to be calm and use QAR plus Beta-t-EGARCH for prediction, after an outlier is observed.

The remainder of this paper is organised as follows. Section II describes the dataset. Section III presents the econometric models. Section IV presents the parameter estimation method. Section V presents the in-sample estimation results. Section VI presents the in-sample forecast performance results. Section VII concludes.

II. Data

We use time-series data for the daily closing value p_t of the S&P 500 market index. The in-sample data window is for period 1950 to 2016. The source of the data is Yahoo Finance, https://finance.yahoo.com/ (accessed on 18th October 2016). We estimate all models for the daily percentage change (i.e. daily return) of the S&P 500, $y_t = (p_t - p_{t-1})/p_{t-1}$ for $t = 1, \dots, T$ days (for p_0, we use pre-sample data for the S&P 500 index). We present the start and end dates of the in-sample window, and sample size T, minimum, maximum, mean, standard deviation, skewness and excess kurtosis of y_t, in Table 1. The excess kurtosis estimate indicates heavy tails for the probability distribution of y_t.

[APPROXIMATE LOCATION OF TABLE 1]

III. Econometric models

First, the AR(p) plus GARCH(1,1) model for the daily S&P 500 returns is

$$y_t = \mu_t + v_t = \mu_t + \lambda_t^{1/2} \epsilon_t \tag{1}$$

$$\mu_t = c + \sum_{j=1}^{p} \phi_j y_{t-j} = c + \sum_{j=1}^{p} \phi_j \left(\mu_{t-j} + \lambda_{t-j}^{1/2} \epsilon_{t-j} \right) \tag{2}$$

$$\lambda_t = \omega + \beta \lambda_{t-1} + \alpha v_{t-1}^2 = \omega + \beta \lambda_{t-1} + \alpha \lambda_{t-1} \epsilon_{t-1}^2 \tag{3}$$

for $t = 1, \ldots, T$, where μ_t is the conditional mean of y_t, v_t is the unexpected return, λ_t is the conditional variance of y_t, and $\epsilon_t \sim N(0,1)$ is the i.i.d. error term representing the new information that arrives to the market. If $S_{\text{Var}} = \alpha + \beta < 1$, then y_t will be covariance stationary in the variance. The initial value of λ_t is estimated by parameter λ_0. The log of the conditional density of y_t is

$$\ln f(y_t | y_1, \ldots, y_{t-1}) = -\frac{1}{2} \ln(2\pi\lambda_t) - \frac{\epsilon_t^2}{2} \tag{4}$$

Second, the QAR(p) plus Beta-t-EGARCH(1,1) model for the daily S&P 500 returns is

$$y_t = \mu_t + v_t = \mu_t + \exp(\lambda_t) \epsilon_t \tag{5}$$

$$\mu_t = c + \sum_{j=1}^{p} \phi_j \mu_{t-j} + \theta e_{t-1} \tag{6}$$

$$\lambda_t = \omega + \beta \lambda_{t-1} + \alpha u_{t-1} \tag{7}$$

for $t = 1, \ldots, T$, where μ_t is the conditional mean of y_t, v_t is the unexpected return, $\exp(\lambda_t)$ is the conditional scale of y_t, and the $\epsilon_t \sim t(\nu)$ i.i.d. error term represents the new information that arrives to the market. If $S_{\text{Var}} = |\beta| < 1$, then y_t will be covariance stationary in the variance.

The initial value of λ_t is estimated by parameter λ_0. The log of the conditional density of y_t is

$$\ln f(y_t|y_1, \ldots, y_{t-1}) = \ln \Gamma\left(\frac{\nu+1}{2}\right) - \ln \Gamma\left(\frac{\nu}{2}\right) - \lambda_t - \frac{\ln(\pi\nu)}{2} - \frac{\nu+1}{2} \ln\left(1 + \frac{\epsilon_t^2}{\nu}\right) \tag{8}$$

where $\Gamma(x)$ is the gamma function. The e_t term in Equation 6 is proportional to the conditional score with respect to μ_t:

$$\frac{\partial \ln f(y_t|y_1, \ldots, y_{t-1})}{\partial \mu_t} = e_t \times \frac{\nu+1}{\nu \exp(2\lambda_t)} = \frac{\nu \exp(\lambda_t)\epsilon_t}{\nu + \epsilon_t^2} \times \frac{\nu+1}{\nu \exp(2\lambda_t)} \tag{9}$$

The u_t term in Equation 7 is the conditional score with respect to λ_t:

$$u_t = \frac{\partial \ln f(y_t|y_1, \ldots, y_{t-1})}{\partial \lambda_t} = \frac{(\nu+1)\epsilon_t^2}{\nu + \epsilon_t^2} - 1 \tag{10}$$

An important difference between AR plus GARCH and QAR plus Beta-t-EGARCH is how μ_t and λ_t are updated after market news arrives. For AR plus GARCH, μ_t and λ_t are updated proportionally to the first lag of ϵ_t (Equation 2) and ϵ_t^2 (Equation 3), respectively. For QAR plus Beta-t-EGARCH, μ_t and λ_t are updated proportionally to the first lag of e_t (Equation 6) and u_t (Equation 7), respectively. We present these updating terms, as functions of ϵ_t, in Fig. 1. The impact of ϵ_t is not discounted in the case of AR due to the linear transformation, and it is accentuated for GARCH due to the quadratic transformation. On the other hand, for QAR plus Beta-t-EGARCH, ϵ_t is discounted for both μ_t and λ_t.

[APPROXIMATE LOCATION OF FIGURE 1]

IV. Parameter estimation

All econometric models in this paper are estimated for the in-sample data window, by using the maximum likelihood (ML) method (Davidson and MacKinnon, 2003). The ML estimator is

$$\hat{\Theta}_{\text{ML}} = \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t|y_1, \ldots, y_{t-1}) \tag{11}$$

where Θ denotes the vector of parameters. We use the robust sandwich ML estimator to compute standard errors of parameters (i.e. robust covariance matrix) (Davidson and MacKinnon, 2003).

V. In-sample estimation results

We present the ML parameter estimates and model diagnostics for AR plus GARCH and QAR plus Beta-t-EGARCH in Table 2. First, in order to identify the lag structure of AR(p) and QAR(p), we perform a preliminary estimation of the partial autocorrelation function (PACF) (Hamilton, 1994) of y_t up to 30 lags. We consider those lags of y_t and μ_t in Equations 2 and 6, respectively, for which the PACF is different from zero, at least at the 10% level of significance. For the initial days of the in-sample data window, we use pre-sample data for the missing values of y_{t-j} and μ_{t-j} (the pre-sample period is from 3rd January 1950 to 14th February 1950). For the ML estimator, we find statistically significant AR and QAR parameters, for several lags (see Table 2). For QAR, we find that θ is positive and significantly different from zero (see Table 2).

Second, we find significant volatility dynamics for both AR plus GARCH and QAR plus Beta-t-EGARCH (Equations 3 and 7, respectively), as α and β are significantly different from zero for both models. Covariance stationarity in the variance is not supported for AR plus GARCH, but it is supported for QAR plus Beta-t-EGARCH (see Table 2).

Third, the degrees of freedom estimate for QAR plus Beta-t-EGARCH suggests heavy tails for y_t, since $\hat{\nu} = 7.1428 < 30$ (see Table 2) (this confirms the excess kurtosis estimate of Table 1).

Fourth, we use the following LL-based model selection metrics: (i) mean LL $= LL/T$; (ii) mean Akaike information criterion (AIC), mean AIC $= 2K/T - 2LL/T$ (K denotes the number of parameters); (iii) mean Bayesian information criterion (BIC), mean BIC $= \ln(T)K/T - 2LL/T$; (iv) mean Hannan-Quinn criterion (HQC), mean HQC $= 2K\ln[\ln(T)]/T - 2LL/T$ (Davidson and MacKinnon, 2003). All metrics suggest that the statistical performance of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH (see Table 2).

Fifth, we use the non-nested likelihood-ratio (LR) test (Vuong, 1989) to study whether the mean LL values of AR plus GARCH and QAR plus Beta-t-EGARCH are significantly different. We define $d_t = \ln f(y_t|y_1,\ldots,y_{t-1}) - \ln g(y_t|y_1,\ldots,y_{t-1})$ for $t = 1,\ldots,T$, where f and g are the conditional density functions of QAR plus Beta-t-EGARCH and AR plus GARCH, respectively. We estimate the linear regression model $d_t = c + \epsilon_t$ for $t = 1, \ldots, T$, by using ordinary least squares (OLS) with heteroscedasticity and autocorrelation consistent (HAC) standard errors (Newey and West, 1987). We find that c is positive and significantly different from zero (see Table 2). Hence, mean LL of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH.

Sixth, we use the Ljung–Box (LB) (1978) test for the residuals $\hat{\epsilon}_t$ with $t = 1, \ldots, T$. Under the null hypothesis of the LB test, ϵ_t for $t = 1, \ldots, T$ are independent. The LB test results support the independence assumption for ϵ_t, for both AR plus GARCH and QAR plus Beta-t-EGARCH (see Table 2). These results suggest that the dynamics used for conditional mean and conditional volatility are effective, for both AR plus GARCH and QAR plus Beta-t-EGARCH.

[APPROXIMATE LOCATION OF TABLE 2]

VI. In-sample forecast performance

For the in-sample forecast performance analysis, we use the following datasets: (D1) all days of the in-sample data window (see Table 1), (D2) each day for which an outlier is observed, and (D3) the trading day after each day for which an outlier is observed. For (D2) and (D3), we define outliers by using Chebyshev's inequality

$$\Pr(|y_t - \mu| \geq k\sigma) \leq \frac{1}{k^2} \quad \text{for} \quad k > 1 \tag{12}$$

where μ and σ are the unconditional mean and unconditional standard deviation of y_t. An advantage of the use of Chebyshev's inequality is that it can be applied to arbitrary probability distributions of y_t with finite μ and σ. μ and σ are estimated by using $\hat{\mu} = \sum_{t=1}^{T} y_t/T$ and $\hat{\sigma} = [\sum_{t=1}^{T}(y_t - \hat{\mu})^2/(T-1)]^{1/2}$, respectively. For the selection of k, we consider the alternatives $k = 3, 4$ and 5, which correspond to 11.11%, 6.25% and 4.00% upper bounds of probability, respectively, in Equation 12. We consider an observation of y_t as an outlier if $|y_t - \hat{\mu}| \geq k\hat{\sigma}$. For $k = 3, 4$ and 5, the number of days with outliers are 235, 97 and 45, respectively, from the in-sample data window (these are the sample sizes for both (D2) and (D3), depending on the choice of k). We present all outliers in the Appendix, where outliers are depicted by $\times$.

For (D1) to (D3), we compare the one-step ahead in-sample forecast performance of AR plus GARCH and QAR plus Beta-t-EGARCH. We study both return and volatility forecasts. The return forecasts for AR plus GARCH and QAR plus Beta-t-EGARCH are $f_{1y,t} = \hat{\mu}_t$ and $f_{2y,t} = \hat{\mu}_t$, respectively. The volatility forecasts for AR plus GARCH and QAR plus Beta-t-EGARCH are $f_{1\sigma,t} = \hat{\lambda}_t^{1/2}$ and $f_{2\sigma,t} = \exp(\hat{\lambda}_t)[\hat{\nu}/(\hat{\nu}-2)]^{1/2}$, respectively. We compare the return and volatility forecasts with y_t (true return) and $|y_t|$ (proxy of true volatility), respectively. The work of Day and Lewis (1992) motivates the use of $|y_t|$ as a proxy of true volatility.

For each day of (D1) to (D3), we measure predictive accuracy by using the Absolute Error (AE) metric. For the return forecasts of AR plus GARCH and QAR plus Beta-t-EGARCH, we use $AE_{1y,t} = |y_t - f_{1y,t}|$ and $AE_{2y,t} = |y_t - f_{2y,t}|$, respectively. For the volatility forecasts of AR plus GARCH and QAR plus Beta-t-EGARCH, we use $AE_{1\sigma,t} = ||y_t| - f_{1\sigma,t}|$ and $AE_{2\sigma,t} = ||y_t| - f_{2\sigma,t}|$, respectively. It is noteworthy that we obtain similar results for the Squared Error (SE) forecast performance metric, e.g. $SE_{1y,t} = (y_t - f_{1y,t})^2$. For each day of (D1) to (D3), we compare AE of AR plus GARCH and QAR plus Beta-t-EGARCH, by using $d_t = AE_{1y,t} - AE_{2y,t}$ (for return forecasting) and $d_t = AE_{1\sigma,t} - AE_{2\sigma,t}$ (for volatility forecasting). For both pairs of AE, we test whether the mean AE (MAE) is significantly different from zero, by using the linear regression model $d_t = c + \epsilon_t$ that is estimated by OLS–HAC. A significantly positive c indicates that the predictive performance of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH. A significantly negative c indicates that the predictive performance of AR plus GARCH is superior to that of QAR plus Beta-t-EGARCH.

We present the estimation results of c in Table 3. First, for the in-sample data window (D1), both return and volatility predictions of QAR plus Beta-t-EGARCH are superior to those of AR plus GARCH (see Table 3). These results suggest that, in general, it is better to be calm and use QAR plus Beta-t-EGARCH, instead of being concerned and using AR plus GARCH, for one-step ahead prediction purposes.

Second, for the days of outliers (D2), the volatility prediction of AR plus GARCH is superior to that of QAR plus Beta-t-EGARCH, for all outlier definitions (see Table 3). We also find that the return predictions of the two models are identical, for all outlier definitions (see Table 3). These results suggest that, for the days with outliers, volatility can be predicted more precisely by being concerned and using AR plus GARCH. According to this result, one would need to know a priori that tomorrow there will be an outlier, and given that information one would use AR plus GARCH for prediction.

Third, for the days after outliers (D3), the return prediction of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH, for all outlier definitions (see Table 3). We also find that the volatility prediction of QAR plus Beta-t-EGARCH is superior to that of AR plus GARCH, for $k = 3$ (see Table 3). Our results for (D3) are useful for practitioners, since they suggest that it is better to be calm and use QAR plus Beta-t-EGARCH, as opposed to being concerned and using AR plus GARCH, after an outlier is observed by the investor.

[APPROXIMATE LOCATION OF TABLE 3]

VII. Conclusion

The manner in which investors react to incoming market news can present a bias in their choice of algorithm as a means of effectively utilising that news. We have studied whether to be concerned or not after news on market value, and use AR plus GARCH or QAR plus Beta-t-EGARCH, respectively. We have used data for period 1950 to 2016 from the S&P 500. We have considered three datasets: all days of the in-sample data window, each day for which an outlier is observed, and the trading day after each day for which an outlier is observed.

For all days of the in-sample data window, our results have suggested that it is better to be calm and use QAR plus Beta-t-EGARCH, instead of being concerned and using AR plus GARCH. For each day for which an outlier is observed, our results have suggested that volatility can be predicted more precisely by being concerned and using AR plus GARCH. According to this result, one would need to know a priori that tomorrow there will be an outlier, and given that information one would use AR plus GARCH for prediction. This result is not very useful for practitioners, since the arrival times of outliers are difficult to predict. For the trading day after each day for which an outlier is observed, our results are useful for practitioners, since they have suggested that it is better to be calm and use QAR plus Beta-t-EGARCH, as opposed to being concerned and using AR plus GARCH.

It is noteworthy that all results reported in this paper are in-sample results. The evaluation of out-of-sample forecasts of AR plus GARCH and QAR plus Beta-t-EGARCH is an extension of the present work and a subject of future research.

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Appendix

Table A1. Outliers

Date	$\lvert y_t - \mu \rvert \geq 3\sigma$	$\lvert y_t - \mu \rvert \geq 4\sigma$	$\lvert y_t - \mu \rvert \geq 5\sigma$	Notes
26-Jun-1950	×	×	×	North Korean troops attack at the South Korean border.
29-Jun-1950	×			North Korean troops attack at the South Korean border.
28-Nov-1950	×			Hearings are opened in the Circuit Court of New York, about the monopoly of investment banks.
4-Dec-1950	×			
9-Feb-1953	×			
6-Jun-1955	×			The post World War II boom
6-Jul-1955	×			The post World War II boom
26-Sep-1955	×	×	×	On 24th September 1955, President Eisenhower had a massive heart attack.
10-Oct-1955	×			
21-Oct-1957	×			
23-Oct-1957	×	×		Suez Canal crisis; the Soviet Union launches Sputnik; the US falls into recession.
17-Apr-1961	×			Bay of Pigs invasion
18-Apr-1961	×			Bay of Pigs invasion
28-May-1962	×	×	×	The Kennedy slide
29-May-1962	×	×		The Kennedy slide
4-Jun-1962	×			The Kennedy slide
28-Jun-1962	×			The Kennedy slide
24-Oct-1962	×			President Kennedy signs the order for naval blockade of Cuba.
26-Nov-1963	×	×		On 22nd November 1963, President John F. Kennedy was assassinated.
27-May-1970	×	×	×	The DJIA increases by 5%. President Richard Nixon called a meeting of leading financial and business leaders in the White House
16-Aug-1971	×			President Richard Nixon ends the convertibility of USD to gold.
24-May-1973	×			The 1973-1974 bear market (dramatic rise in oil prices, the miners' strike and the downfall of the Heath government in the UK).
19-Nov-1973	×			The 1973-1974 bear market
26-Nov-1973	×			The 1973-1974 bear market
26-Dec-1973	×			The 1973-1974 bear market

Table A2. Outliers

Date	$\lvert y_t - \mu\rvert \geq 3\sigma$	$\lvert y_t - \mu\rvert \geq 4\sigma$	$\lvert y_t - \mu\rvert \geq 5\sigma$	Notes
8-Jul-1974	×			The 1973-1974 bear market
12-Jul-1974	×	×		The 1973-1974 bear market
30-Aug-1974	×			The 1973-1974 bear market
5-Sep-1974	×			The 1973-1974 bear market
19-Sep-1974	×			The 1973-1974 bear market
7-Oct-1974	×	×		The 1973-1974 bear market
9-Oct-1974	×	×		The 1973-1974 bear market
23-Oct-1974	×			The 1973-1974 bear market
29-Oct-1974	×	×		The 1973-1974 bear market
18-Nov-1974	×			The 1973-1974 bear market
27-Jan-1975	×			The 1973-1974 bear market
1-Nov-1978	×	×		
9-Oct-1979	×			
17-Mar-1980	×			Stock market bubble at the Souk Al-Manakh exchange in Kuwait
24-Mar-1980	×			Stock market bubble at the Souk Al-Manakh exchange in Kuwait
22-Apr-1980	×			Stock market bubble at the Souk Al-Manakh exchange in Kuwait
24-Aug-1981	×			
17-Aug-1982	×	×		Souk Al-Manakh stock market crash
20-Aug-1982	×			Souk Al-Manakh stock market crash
6-Oct-1982	×			
25-Oct-1982	×	×		
3-Nov-1982	×	×		
30-Nov-1982	×			
7-Jul-1986	×			
11-Sep-1986	×	×	×	
14-Oct-1987	×			
16-Oct-1987	×	×	×	
19-Oct-1987	×	×	×	Black Monday
20-Oct-1987	×	×	×	Black Monday aftermath
21-Oct-1987	×	×	×	Black Monday aftermath

Table A3. Outliers

Date	$\lvert y_t - \mu \rvert \geq 3\sigma$	$\lvert y_t - \mu \rvert \geq 4\sigma$	$\lvert y_t - \mu \rvert \geq 5\sigma$	Notes
22-Oct-1987	×	×		Black Monday aftermath
26-Oct-1987	×	×	×	Black Monday aftermath
29-Oct-1987	×	×	×	Black Tuesday
9-Nov-1987	×			
30-Nov-1987	×	×		
3-Dec-1987	×			
4-Jan-1988	×			
8-Jan-1988	×	×	×	
14-Apr-1988	×	×		The DJIA declined by more than 50 points.
31-May-1988	×			
13-Oct-1989	×	×	×	Friday the 13th mini-crash
6-Aug-1990	×			Gulf War, United Nations (UN) imposes sanctions on Iraq.
23-Aug-1990	×			East Germany and West Germany announced that they would unite on 3rd October 1990. Gulf War, US begins the call-up of 46,000 reservists to the Persian Gulf.
27-Aug-1990	×			Gulf War, oil market prices plunge as OPEC reaches an informal agreement to increase output in order to cover shortfall due to invasion.
17-Jan-1991	×			Gulf War, Operation Desert Storm begins against Iraq.
21-Aug-1991	×			Conservative coup in the Soviet Union is defeated by the popular resistance led by Boris Yeltsin. Latvia declares its independence from the Soviet Union.
15-Nov-1991	×			Bad economic statistics cause fear of economic stagnation.
8-Mar-1996	×			Unexpectedly promising unemployment report.
2-Sep-1997	×			Asian financial and economic crisis
27-Oct-1997	×	×	×	Asian financial and economic crisis
28-Oct-1997	×	×	×	Asian financial and economic crisis
9-Jan-1998	×			Asian financial and economic crisis
4-Aug-1998	×			Asian financial and economic crisis. A slower US economic growth and lower corporate profit forecasts for the rest of the year. Worst day for stocks during 1998.
27-Aug-1998	×	×		Russian financial crisis
31-Aug-1998	×	×	×	Russian financial crisis and a slower US economic growth.
1-Sep-1998	×			Russian financial crisis

Table A4. Outliers

Date	$\lvert y_t-\mu\rvert \geq 3\sigma$	$\lvert y_t-\mu\rvert \geq 4\sigma$	$\lvert y_t-\mu\rvert \geq 5\sigma$	Notes
8-Sep-1998	×	×	×	FED Chairman Alan Greenspan gives hopes that FED would ease interest rates.
11-Sep-1998	×			IMF announces that the fall in Latin American markets is an overreaction to Russian events, and that it is ready to lend to Latin American countries by using an emergency line of credit. Investors flee from Brazil, drawing out more than USD2 billion per day (despite the 50% interest rate rise by the Central Bank of Brazil).
23-Sep-1998	×			Russian financial crisis. Motivated by the New York FED, a consortium of leading US financial institutions provides a USD3.5 billion bailout to Long-Term Capital Management (LTCM).
30-Sep-1998	×			Worries that FED is not doing enough to support the US and global economic growth cause a 238-point drop in the DJIA, for a loss of more than 500 points in a week. Investors around the world flee to US Treasury-bonds for safety, causing the yield on 30-year bonds to drop below 5% for the first time in three decades.
1-Oct-1998	×			
15-Oct-1998	×	×		FED cuts interest rates for a second time to prevent weak financial markets from driving the US into a recession. The DJIA increases by 331 points and world market prices also increase.
28-Oct-1999	×			Brazilian crisis
4-Jan-2000	×			Investors are considering the possibility of higher interest rates.
18-Feb-2000	×			Fears about higher interest rates affect blue-chip firms and technology stocks.
16-Mar-2000	×	×		Blue-chip firms have a strong demand. The DJIA increases by 499.19 points. New York Stock Exchange (NYSE) has a very busy trading day.
14-Apr-2000	×	×	×	Inflationary fears
17-Apr-2000	×			
25-Apr-2000	×			
30-May-2000	×			Best day of the history of National Association of Securities Dealers Automated Quotations (NASDAQ) index.
13-Oct-2000	×			A large volume of cheap stock purchase increases NASDAQ by 8%.
19-Oct-2000	×			Strong earnings from several technology firms. The third biggest gain of the history in the NASDAQ index.

Table A5. Outliers

Date	$\lvert y_t - \mu \rvert \geq 3\sigma$	$\lvert y_t - \mu \rvert \geq 4\sigma$	$\lvert y_t - \mu \rvert \geq 5\sigma$	Notes
5-Dec-2000	×			Microsoft announces that it would not achieve the profit forecast for the first time in a decade. The DJIA decreases by more than 240 points.
20-Dec-2000	×			Selling hysteria in the technology sector.
3-Jan-2001	×	×	×	Interest rate cut by the FED. NASDAQ increases by 14%.
12-Mar-2001	×	×		New worries that the slowing US economy has not finished reducing corporate profits.
3-Apr-2001	×			Fears about profit growth cause triple-digit losses in the DJIA and NASDAQ.
5-Apr-2001	×	×		Good news from Alcoa and Dell Computer.
18-Apr-2001	×			Interest rate cut by the FED
17-Sep-2001	×	×	×	September 11 attacks
20-Sep-2001	×			Growing concerns over what direction the US will take after the September 11 attacks.
24-Sep-2001	×			Positive analyst and corporate comments.
8-May-2002	×			Cisco increases NASDAQ by 7%; high after earnings forecasts for networking-gear makers; the DJIA is above 10,000 points.
5-Jul-2002	×			
10-Jul-2002	×			Accounting scandals and low forecasts of earnings growth decrease the S&P 500 and NASDAQ to their lowest levels since 1997.
19-Jul-2002	×			Drastic loss of confidence in the stock markets in the US. The DJIA hits its lowest level in nearly four years.
22-Jul-2002	×			Concerns about the Enron connections of Citigroup, and low quarterly results from BellSouth.
24-Jul-2002	×	×	×	Investors returned at full-strength to buy after weeks of almost non-stop selling.
29-Jul-2002	×	×	×	Investors returned at full-strength to buy after weeks of almost non-stop selling.
1-Aug-2002	×			Institute of Supply Management (ISM) manufacturing index results are published, showing markets that manufacturing is slowing.
5-Aug-2002	×			Stock prices decrease before the FED interest rate policy decision.
6-Aug-2002	×			Stock prices decrease before the FED interest rate policy decision.
8-Aug-2002	×			Stock prices increase as International Monetary Fund (IMF) provides a USD30 billion loan guarantee for Brazil.

Table A6. Outliers

Date	$\lvert y_t - \mu \rvert \geq 3\sigma$	$\lvert y_t - \mu \rvert \geq 4\sigma$	$\lvert y_t - \mu \rvert \geq 5\sigma$	Notes
14-Aug-2002	×	×		
3-Sep-2002	×	×		During the Labor Day Weekend, Consolidated Freightways filed bankruptcy and sacked 15,000 employees.
19-Sep-2002	×			OPEC agrees to leave output unchanged. Both Pepsi and Coca-Cola are downgraded by UBS.
27-Sep-2002	×			The ratings of General Electric (GE) are reduced by several analysts overnight. The final revision of the second quarter GDP is announced with a 1.3% increase.
1-Oct-2002	×	×		The Institute of Supply Management (ISM) manufacturing index was lower than expected.
10-Oct-2002	×			Stock prices increased sharply from depressed levels. A solid profit report and forecast from Yahoo! helps to increase stock prices in the technology business sector.
11-Oct-2002	×	×		Retail sales decrease by 1.2%; Producer Price Index (PPI) increases by 0.1%; stock futures prices increase; Jimmy Carter wins the Nobel Peace Prize.
15-Oct-2002	×	×		Optimism overcomes fears of more terror attacks sparked by the bombing in Bali.
2-Jan-2003	×			JPMorgan (JPM) stock prices are influenced by the news of a settlement in a law suit against insurers, in which JPM seeks USD1 billion to cover losses in Enron.
24-Jan-2003	×			Utility crisis shown by CMS Energy, due to no dividend payments. USD weakens as China and Russia have announced that they want to hold fewer reserves in USD.
13-Mar-2003	×			Stock prices increase, as several market gurus announce that the market has reached its lowest value.
17-Mar-2003	×			Saint Patrick's day. European and Asian markets are down due to worries about wars and killings shown on the news.
24-Mar-2003	×			Stock prices decrease, and there are worries about the war in Baghdad.
27-Feb-2007	×			The DJIA decreases by 410 points.
9-Aug-2007	×			FED Chairman Ben Bernanke talks about the markets, and gives hope that FED will address the mortgage problem.
7-Nov-2007	×			China releases news about the diversification of its USD investments, after the USD has significantly fallen against the euro.
17-Jan-2008	×			Subprime mortgage crisis
5-Feb-2008	×			Subprime mortgage crisis

Table A7. Outliers

Date	$\lvert y_t - \mu\rvert \geq 3\sigma$	$\lvert y_t - \mu\rvert \geq 4\sigma$	$\lvert y_t - \mu\rvert \geq 5\sigma$	Notes
11-Mar-2008	×			Subprime mortgage crisis
18-Mar-2008	×	×		Subprime mortgage crisis
1-Apr-2008	×			Subprime mortgage crisis
6-Jun-2008	×			Subprime mortgage crisis
26-Jun-2008	×			Subprime mortgage crisis
4-Sep-2008	×			Subprime mortgage crisis
9-Sep-2008	×			Subprime mortgage crisis
15-Sep-2008	×	×		Subprime mortgage crisis
17-Sep-2008	×	×		Subprime mortgage crisis
18-Sep-2008	×	×		Subprime mortgage crisis
19-Sep-2008	×	×		Subprime mortgage crisis
22-Sep-2008	×			Subprime mortgage crisis
29-Sep-2008	×	×	×	The US House of Representatives rejects a USD700 billion bank bailout plan.
30-Sep-2008	×	×	×	Subprime mortgage crisis
2-Oct-2008	×	×		Subprime mortgage crisis
6-Oct-2008	×	×		Subprime mortgage crisis
7-Oct-2008	×	×	×	Subprime mortgage crisis
9-Oct-2008	×	×	×	Subprime mortgage crisis
13-Oct-2008	×	×	×	Subprime mortgage crisis
15-Oct-2008	×	×	×	FED Chairman Ben Bernanke says that the economic recovery will be slow.
16-Oct-2008	×	×		Subprime mortgage crisis
20-Oct-2008	×	×		Subprime mortgage crisis
21-Oct-2008	×			Subprime mortgage crisis
22-Oct-2008	×	×	×	Subprime mortgage crisis
24-Oct-2008	×			Subprime mortgage crisis
27-Oct-2008	×			Subprime mortgage crisis
28-Oct-2008	×	×	×	Subprime mortgage crisis
4-Nov-2008	×	×		Subprime mortgage crisis
5-Nov-2008	×	×	×	Subprime mortgage crisis
6-Nov-2008	×	×	×	Subprime mortgage crisis

Table A8. Outliers

Date	$\lvert y_t - \mu \rvert \geq 3\sigma$	$\lvert y_t - \mu \rvert \geq 4\sigma$	$\lvert y_t - \mu \rvert \geq 5\sigma$	Notes
12-Nov-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
13-Nov-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
14-Nov-2008	$\times$	$\times$		Subprime mortgage crisis
19-Nov-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
20-Nov-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
21-Nov-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
24-Nov-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
26-Nov-2008	$\times$			Subprime mortgage crisis
1-Dec-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
2-Dec-2008	$\times$	$\times$		Subprime mortgage crisis
4-Dec-2008	$\times$			Subprime mortgage crisis
5-Dec-2008	$\times$			Subprime mortgage crisis
8-Dec-2008	$\times$			Subprime mortgage crisis
16-Dec-2008	$\times$	$\times$	$\times$	Subprime mortgage crisis
2-Jan-2009	$\times$			Subprime mortgage crisis
7-Jan-2009	$\times$			Subprime mortgage crisis
14-Jan-2009	$\times$			Subprime mortgage crisis
20-Jan-2009	$\times$	$\times$	$\times$	Subprime mortgage crisis
21-Jan-2009	$\times$	$\times$		Subprime mortgage crisis
28-Jan-2009	$\times$			Subprime mortgage crisis
29-Jan-2009	$\times$			Subprime mortgage crisis
10-Feb-2009	$\times$	$\times$	$\times$	Subprime mortgage crisis
17-Feb-2009	$\times$	$\times$		Subprime mortgage crisis
23-Feb-2009	$\times$			Subprime mortgage crisis
24-Feb-2009	$\times$	$\times$		Subprime mortgage crisis
2-Mar-2009	$\times$	$\times$		Subprime mortgage crisis
5-Mar-2009	$\times$	$\times$		Subprime mortgage crisis
10-Mar-2009	$\times$	$\times$	$\times$	Subprime mortgage crisis
12-Mar-2009	$\times$	$\times$		Subprime mortgage crisis
17-Mar-2009	$\times$			Subprime mortgage crisis

Table A9. Outliers

Date	$\lvert y_t - \mu \rvert \geq 3\sigma$	$\lvert y_t - \mu \rvert \geq 4\sigma$	$\lvert y_t - \mu \rvert \geq 5\sigma$	Notes
23-Mar-2009	×	×	×	Subprime mortgage crisis
30-Mar-2009	×			Subprime mortgage crisis
9-Apr-2009	×			Subprime mortgage crisis
20-Apr-2009	×	×		Subprime mortgage crisis
4-May-2009	×			Subprime mortgage crisis
18-May-2009	×			Subprime mortgage crisis
22-Jun-2009	×			Subprime mortgage crisis
2-Jul-2009	×			
15-Jul-2009	×			FED releases news that it sees the recession ending and expects growth in the economy for the coming year, while unemployment continues to grow.
4-Feb-2010	×			Portugal and Spain lead the worldwide decline in stock market, due to their budget deficit and spending cuts.
6-May-2010	×			
10-May-2010	×	×		
20-May-2010	×	×		European stock prices decrease, as sentiment continues to be negative by concerns about the European debt crisis. A news release shows an unexpected increase in US unemployment claims.
27-May-2010	×			Jobless claims in the US are at 460,000; China releases news that it will continue to buy European bonds.
4-Jun-2010	×			Private payrolls added only 24,000 jobs in the US (less than expected).
10-Jun-2010	×			European stock prices increase due to the strong Chinese export data; a successful Spanish bond auction is organised; the European Central Bank (ECB) releases optimistic news.
29-Jun-2010	×			
7-Jul-2010	×			US stocks react to the decline in Asian stocks. The weak US data cause worries about the global economic recovery.
16-Jul-2010	×			General Electric (GE) has a profit gain in the first quarter, showing that the economy might be in recovery since the financial crisis.
1-Sep-2010	×			
4-Aug-2011	×	×		The DJIA decreases by 512 points, the ninth deepest point drop in the history.

Table A10. Outliers

Date	$\lvert y_t - \mu \rvert \geq 3\sigma$	$\lvert y_t - \mu \rvert \geq 4\sigma$	$\lvert y_t - \mu \rvert \geq 5\sigma$	Notes
8-Aug-2011	×	×	×	Fears about a global slowdown
9-Aug-2011	×	×		European debt crisis
10-Aug-2011	×	×		European debt crisis
11-Aug-2011	×	×		European debt crisis
18-Aug-2011	×	×		European debt crisis
23-Aug-2011	×			European debt crisis
21-Sep-2011	×			European debt crisis
22-Sep-2011	×			European debt crisis
10-Oct-2011	×			European debt crisis
27-Oct-2011	×			The European Union (EU) reaches an agreement, and stock markets increase.
9-Nov-2011	×			Italian bond crisis
30-Nov-2011	×	×		European debt crisis
20-Dec-2011	×			The EU is showing positive signs. US housing starts with strong numbers and market prices increase.
21-Aug-2015	×			Stock prices decrease by 6% in China, which causes worries in US markets.
24-Aug-2015	×	×		Markets are concerned about economic growth in China.
26-Aug-2015	×			Markets are concerned about economic growth in China.
1-Sep-2015	×			Markets are concerned that FED is going to increase interest rates to 0.25%.
24-Jun-2016	×			On 23rd June 2016, the UK voted to leave the EU via referendum.

Table 1. Descriptive statistics of S&P 500 return, in-sample data window (D1)

Start date	2nd February 1950
End date	17th October 2016
Sample size T	$16,777$
Minimum	-0.2047
Maximum	0.1158
Mean	0.0003
Standard deviation	0.0097
Skewness	-0.6402
Excess kurtosis	20.7854

Table 2. Parameter estimates and model diagnostics, in-sample data window (D1)

	AR plus GARCH	QAR plus Beta-t-EGARCH
c	$0.0005^{***}(0.0001)$	$0.0009^{***}(0.0003)$
ϕ_1	$0.0926^{***}(0.0086)$	$-0.2327^{***}(0.0763)$
ϕ_2	$-0.0258^{***}(0.0085)$	$-0.1510^{*}(0.0796)$
ϕ_5	$-0.0099(0.0084)$	$-0.1810^{**}(0.0913)$
ϕ_6	$-0.0198^{**}(0.0086)$	$-0.1534(0.1204)$
ϕ_7	$-0.0107(0.0081)$	$-0.0829(0.0982)$
ϕ_9	$-0.0101(0.0088)$	$0.0900(0.0617)$
ϕ_{10}	$0.0132(0.0080)$	$-0.1239(0.0820)$
ϕ_{11}	$-0.0137^{*}(0.0083)$	$0.0771(0.0868)$
ϕ_{12}	$0.0111(0.0077)$	$0.1203(0.0931)$
ϕ_{15}	$-0.0084(0.0080)$	$0.1055(0.0935)$
ϕ_{16}	$0.0135^{*}(0.0079)$	$0.0553(0.0759)$
ϕ_{18}	$-0.0086(0.0081)$	$-0.0849(0.0602)$
ϕ_{21}	$-0.0179^{**}(0.0079)$	$0.0363(0.0911)$
ϕ_{24}	$0.0141^{*}(0.0082)$	$-0.1425^{**}(0.0557)$
ϕ_{25}	$-0.0218^{***}(0.0081)$	$-0.1011^{*}(0.0608)$
ϕ_{26}	$-0.0181^{**}(0.0076)$	$-0.0283(0.1008)$
ϕ_{27}	$0.0096(0.0082)$	$-0.0326(0.0737)$
ϕ_{29}	$0.0100(0.0078)$	$0.1481^{*}(0.0766)$
θ	NA	$0.1371^{***}(0.0112)$
ω	$0.0000^{***}(0.0000)$	$-0.0534^{***}(0.0082)$
α	$0.0861^{***}(0.0111)$	$0.0442^{***}(0.0030)$
β	$0.9057^{***}(0.0112)$	$0.9893^{***}(0.0016)$
λ_0	$0.0000(0.0000)$	$-5.5522^{***}(0.3265)$
ν	NA	$7.1428^{***}(0.4327)$
S_{Var}	0.9919	0.9893
mean LL	3.4085	3.4357
mean AIC	−6.8142	−6.8684
mean BIC	−6.8036	−6.8569
mean HQC	−6.8107	−6.8646
c for $d_t = c + \epsilon_t$	NA	$0.0273^{***}(0.0051)$
LB statistic	24.2383	23.1951
LB p-value	0.9767	0.9845

Notes: Autoregressive (AR); generalized autoregressive conditional heteroscedasticity (GARCH); quasi-AR (QAR); exponential GARCH (EGARCH); not available (NA); log-likelihood (LL); Akaike information criterion (AIC); Bayesian information criterion (BIC); Hannan–Quinn criterion (HQC); Ljung–Box (LB). S_{Var} is the covariance stationarity in the variance statistic. $d_t = c + \epsilon_t$ is estimated by using ordinary least squares (OLS) with heteroscedasticity and autocorrelation consistent (HAC) standard error. Robust standard errors are shown in parentheses. *, ** and *** indicate parameter significance at the 10%, 5% and 1% levels, respectively.

Table 3. Forecast performance, OLS–HAC estimates of c for the linear regression $d_t = c + \epsilon_t$

(D1) All days of the in-sample data window:		
Expected return	8.8061E-06**(4.1621E-06)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	5.0938E-05***(8.1275E-06)	QAR-Beta-t-EGARCH is superior to AR-GARCH
(D2) Each day for which an outlier is observed ($\|y_t - \mu\| \geq 3\sigma$):		
Expected return	1.4198E-04(1.1589E-04)	Models are identical
Volatility	-1.2971E-03***(2.8109E-04)	AR-GARCH is superior to QAR-Beta-t-EGARCH
(D2) Each day for which an outlier is observed ($\|y_t - \mu\| \geq 4\sigma$):		
Expected return	2.6067E-04(2.5278E-04)	Models are identical
Volatility	-2.6565E-03***(5.9441E-04)	AR-GARCH is superior to QAR-Beta-t-EGARCH
(D2) Each day for which an outlier is observed ($\|y_t - \mu\| \geq 5\sigma$):		
Expected return	4.0602E-04(4.8409E-04)	Models are identical
Volatility	-4.4296E-03***(1.0427E-03)	AR-GARCH is superior to QAR-Beta-t-EGARCH
(D3) The trading day after each day for which an outlier is observed ($\|y_t - \mu\| \geq 3\sigma$):		
Expected return	5.3342E-04***(1.4761E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	7.5792E-04**(3.4555E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
(D3) The trading day after each day for which an outlier is observed ($\|y_t - \mu\| \geq 4\sigma$):		
Expected return	1.0119E-03***(3.0816E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	9.6337E-04(7.8732E-04)	Models are identical
(D3) The trading day after each day for which an outlier is observed ($\|y_t - \mu\| \geq 5\sigma$):		
Expected return	1.5911E-03***(5.8619E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	4.0287E-04(1.4763E-03)	Models are identical

Notes: Ordinary least squares (OLS); heteroscedasticity and autocorrelation consistent (HAC); autoregressive (AR); generalized autoregressive conditional heteroscedasticity (GARCH); quasi-AR (QAR); exponential GARCH (EGARCH). μ and σ denote the unconditional mean and unconditional standard deviation, respectively, of y_t. Robust standard errors are shown in parentheses. ** and *** indicate parameter significance at the 5% and 1% levels, respectively.

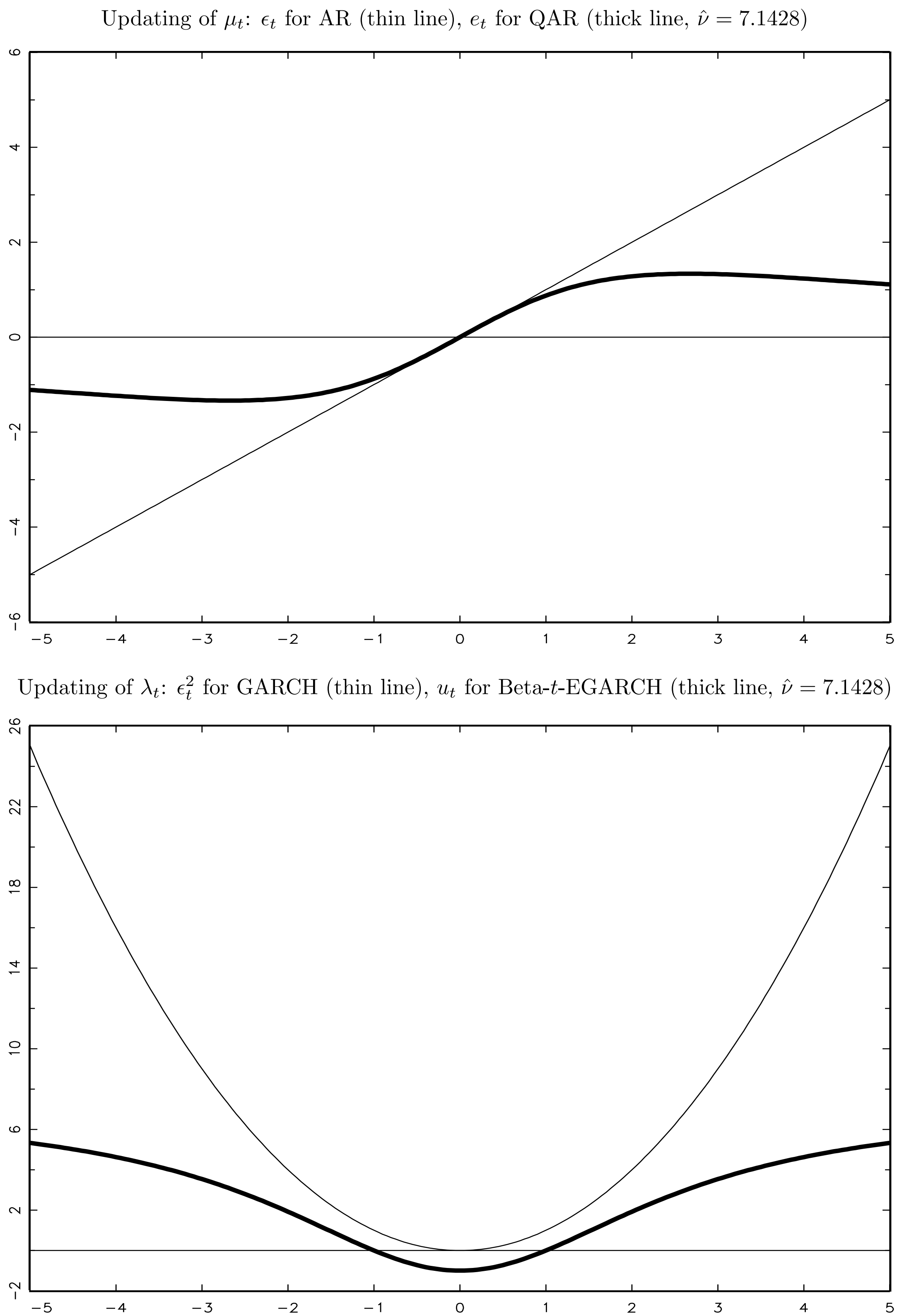

Fig. 1. Updating of μ_t and λ_t after news, as a function of ϵ_t.

Chapter 2
Forecasting following appearance of extreme values when using AR-GARCH and QAR-Beta-t-EGARCH

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Abstract: We undertake a systematic review of the return and volatility predictive performances of the standard AR-GARCH and the recent QAR-Beta-t-EGARCH models. We use historical data from the Dow Jones Industrial Average (DJIA) index for the hundred-year period of May 1896 to March 2017. We compare predictive performances for those days when extreme value is observed, and also for the trading day after each day when extreme value is observed. We use alternative definitions of extreme values, according to the Chebyshev inequality. We find that AR-GARCH dominates QAR-Beta-t-EGARCH for each day for which an extreme value is observed, and QAR-Beta-t-EGARCH dominates AR-GARCH for the trading day after each day for which an extreme value is observed.

Keywords: dynamic conditional score (DCS) models; QAR (quasi-autoregressive) model; Beta-t-EGARCH model; extreme values

JEL classification: C22, C52, C58

I. Introduction

Harvey (2013, p. 133) presents an application of GARCH (generalized autoregressive conditional heteroscedasticity) (Bollerslev 1986; Taylor 1986) and Beta-t-EGARCH (exponential GARCH) (Harvey and Chakravarty 2008), both with leverage effects, for the Dow Jones Industrial Average (DJIA) index. Harvey (2013) uses data for period October 1975 to August 2009, including Black Monday (19th October 1987, when DJIA declined 22.61%). Harvey (2013) notes that the conditional volatility estimates for GARCH and Beta-t-EGARCH exhibit a marked difference after the appearance of extreme values.

Motivated by Harvey (2013), we undertake a systematic review of the return and volatility forecast performances of AR (autoregressive) (Box and Jenkins 1970) plus GARCH and QAR (quasi-AR) (Harvey 2013) plus Beta-t-EGARCH, for the DJIA index. We study forecast performances for those days when extreme value is observed, and also for the trading day after each day when extreme value is observed.

II. Data

We use historical time-series data for the daily closing value p_t of DJIA for period May 1896 to March 2017 (source: S&P Dow Jones Indices, http://www.djaverages.com, accessed 12th March 2017). We estimate all models for the daily log-return $y_t = \ln(p_t/p_{t-1})$ for days $t = 1, \ldots, T$ (for p_0, we use pre-sample data). We present some descriptive statistics of y_t in Table 1.

[APPROXIMATE LOCATION OF TABLE 1]

III. Econometric models

Firstly, the AR(p)-GARCH(1,1) model with leverage effects is

$$y_t = \mu_t + v_t = \mu_t + \lambda_t^{1/2}\epsilon_t \text{ with } \epsilon_t \sim N(0,1) \text{ i.i.d.} \tag{13}$$

$$\mu_t = c + \sum_{j=1}^{p} \phi_j y_{t-j} = c + \sum_{j=1}^{p} \phi_j \left(\mu_{t-j} + \lambda_{t-j}^{1/2}\epsilon_{t-j}\right) \tag{14}$$

$$\lambda_t = \omega + [\alpha + \alpha^* \mathbb{1}(v_{t-1} < 0)]v_{t-1}^2 + \beta\lambda_{t-1} = \omega + [\alpha + \alpha^* \mathbb{1}(\epsilon_{t-1} < 0)]\lambda_t\epsilon_t^2 + \beta\lambda_{t-1} \tag{15}$$

for $t = 1, \ldots, T$, where $\mathbb{1}(\cdot)$ is the indicator function. The conditional mean and volatility of y_t are μ_t and $\lambda_t^{1/2}$, respectively. The initial value of λ_t is estimated by parameter λ_0.

Secondly, the QAR(p)-Beta-t-EGARCH(1,1) model with leverage effects is

$$y_t = \mu_t + v_t = \mu_t + \exp(\lambda_t)\epsilon_t \text{ with } \epsilon_t \sim t(\nu) \text{ i.i.d.} \tag{16}$$

$$\mu_t = c + \sum_{j=1}^{p} \phi_j \mu_{t-j} + \theta e_{t-1} = c + \sum_{j=1}^{p} \phi_j \mu_{t-j} + \theta \left[\frac{\nu \exp(\lambda_{t-1})\epsilon_{t-1}}{\nu + \epsilon_{t-1}^2} \right] \tag{17}$$

$$\lambda_t = \omega + \alpha u_{t-1} + \alpha^* \text{sgn}(-v_{t-1})(u_{t-1} + 1) + \beta \lambda_{t-1} \tag{18}$$

for $t = 1, \ldots, T$, where $\text{sgn}(\cdot)$ is the signum function and $u_t = [(\nu + 1)\epsilon_t^2]/[\nu + \epsilon_t^2] - 1$. The conditional mean and volatility of y_t are μ_t and $\exp(\lambda_t)[\nu/(\nu - 2)]^{1/2}$, respectively. The initial value of λ_t is estimated by parameter λ_0.

The marked difference between the conditional volatility estimates of GARCH and Beta-t-EGARCH (Harvey 2013, p. 133), is due to the way in which λ_t is updated after market news arrives. We present the updating terms of AR-GARCH and QAR-Beta-t-EGARCH as functions of ϵ_t, in Fig. 1. The impact of ϵ_t is not discounted in the case of AR due to the linear transformation, and it is accentuated for GARCH due to the quadratic transformation. On the other hand, for QAR-Beta-t-EGARCH, ϵ_t is discounted for both μ_t and λ_t.

[APPROXIMATE LOCATION OF FIGURE 1]

IV. Statistical inference

All models in this paper are estimated by using the maximum likelihood (ML) method (Davidson and MacKinnon 2003). The ML estimator is

$$\hat{\Theta}_{\text{ML}} = \arg\max_{\Theta} \text{LL} = \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t | y_1, \ldots, y_{t-1}) \tag{19}$$

where Θ is the vector of time-constant parameters and LL is log-likelihood. We use the sandwich covariance matrix estimator to compute robust standard errors of parameters.

We focus on the conditions of the Gaussian central limit theory (GCLT) of the ML estimator for λ_t (we assume that GCLT conditions for μ_t are satisfied for both the AR(p) and QAR(p)). The GCLT conditions for GARCH with leverage effects hold if (Jensen and Rahbek 2004) $\text{GCLT}_\lambda = E\{\beta/[(\alpha + \alpha^*/2)\epsilon_t + \beta]\} < 1$ (we estimate this expectation by the sample average). The GCLT conditions for Beta-t-EGARCH with leverage effects hold if (Harvey 2013):

$$\text{GCLT}_\lambda = \beta^2 - \alpha\beta\frac{4\nu}{\nu+3} + [\alpha^2 + (\alpha^*)^2]\frac{12\nu(\nu+1)(\nu+2)}{(\nu+7)(\nu+5)(\nu+3)} < 1 \tag{20}$$

V. Estimation results

Firstly, in order to identify the lag structure of AR(p) and QAR(p), we estimate the partial autocorrelation function (PACF) (Hamilton 1994) of y_t up to 30 lags. We consider those lags only, for which PACF is different from zero, at least at the 10% level of significance. For the initial days of the dataset, we use pre-sample data for y_{t-j} and μ_{t-j} (the 30-day pre-sample period is from 26th May 1896 to 1st July 1896). In Table 1, we present the ML estimates and model diagnostics for AR-GARCH and QAR-Beta-t-EGARCH. We find significant ϕ_j parameters for several lags; we find that θ is significant for QAR; we find that α, α^* and β are all significantly different from zero for both GARCH and Beta-t-EGARCH.

Secondly, we use the following statistical performance metrics: (i) mean LL = LL$/T$; (ii) mean Akaike information criterion (AIC), mean AIC $= 2K/T - 2\text{LL}/T$ (K denotes the number of parameters); (iii) mean Bayesian information criterion (BIC), mean BIC $= \ln(T)K/T - 2\text{LL}/T$; (iv) mean Hannan-Quinn criterion (HQC), mean HQC $= 2K\ln[\ln(T)]/T - 2\text{LL}/T$. All metrics suggest that QAR-Beta-t-EGARCH is superior to AR-GARCH.

Thirdly, we use the non-nested likelihood-ratio (LR) test (Vuong 1989) to study whether the mean LL values of AR-GARCH and QAR-Beta-t-EGARCH are significantly different. We define $d_t = \ln f(y_t|y_1,\ldots,y_{t-1}) - \ln g(y_t|y_1,\ldots,y_{t-1})$ for $t = 1,\ldots,T$, where f and g are the conditional density functions of QAR-Beta-t-EGARCH and AR-GARCH, respectively. We estimate the linear regression model $d_t = c + \epsilon_t$ for $t = 1,\ldots,T$, by using ordinary least squares (OLS) with heteroscedasticity and autocorrelation consistent (HAC) standard errors (Newey and West 1987).

For the estimate of c, we find $0.0304^{***}(0.0041)$, i.e. c is positive and significantly different from zero. Hence, mean LL of QAR-Beta-t-EGARCH is superior to that of AR-GARCH.

Fourthly, we use the Ljung–Box (1978) test (hereafter, LB test) with the lag order 30 for the residual time-series $(\hat{\epsilon}_1, \ldots, \hat{\epsilon}_T)$. Under the null hypothesis of the LB test, $(\epsilon_1, \ldots, \epsilon_T)$ are independent. We find that this null hypothesis is supported for both AR-GARCH and QAR-Beta-t-EGARCH.

VI. Predictive performance for extreme values

We use data for: (D1) each day on which an extreme value is observed, and (D2) the trading day after each day on which an extreme value is observed. We define extreme values by using the Chebyshev inequality

$$\Pr(|y_t - \mu| \geq k\sigma) \leq \frac{1}{k^2} \quad \text{for} \quad k > 1 \tag{21}$$

where μ and σ are the unconditional mean and unconditional standard deviation of y_t, respectively. We estimate μ and σ by using $\hat{\mu} = \sum_{t=1}^{T} y_t/T$ and $\hat{\sigma} = [\sum_{t=1}^{T}(y_t - \hat{\mu})^2/(T-1)]^{1/2}$, respectively. For the selection of k, we consider the alternatives $k = 3, 4, 5$ and 6, which correspond to 11.11%, 6.25%, 4.00% and 2.78% upper bounds of probability, respectively, in Equation (9). We consider an observation of y_t as an extreme value if $|y_t - \hat{\mu}| \geq k\hat{\sigma}$. For $k = 3, 4, 5$ and 6, the number of days with extreme values are 516 (1.57%), 224 (0.68%), 100 (0.30%) and 52 (0.15%), respectively, from $T = 32,865$ days (100%).

The one-step ahead return forecasts for AR-GARCH and QAR-Beta-t-EGARCH are $f_{1y,t} = \hat{\mu}_t$ and $f_{2y,t} = \hat{\mu}_t$, respectively. The one-step ahead volatility forecasts for AR-GARCH and QAR-Beta-t-EGARCH are $f_{1\sigma,t} = \hat{\lambda}_t^{1/2}$ and $f_{2\sigma,t} = \exp(\hat{\lambda}_t)[\hat{\nu}/(\hat{\nu} - 2)]^{1/2}$, respectively. We compare the return forecasts with y_t. We compare the volatility forecasts with $|y_t|$ (the work of Day and Lewis [1992] motivates the use of $|y_t|$ as a proxy of true volatility).

For each day, we measure predictive accuracy by using the Absolute Error (AE) metric. For the return forecasts of AR-GARCH and QAR-Beta-t-EGARCH, we use $AE_{1y,t} = |y_t - f_{1y,t}|$ and

$AE_{2y,t} = |y_t - f_{2y,t}|$, respectively. For the volatility forecasts of AR-GARCH and QAR-Beta-t-EGARCH, we use $AE_{1\sigma,t} = ||y_t| - f_{1\sigma,t}|$ and $AE_{2\sigma,t} = ||y_t| - f_{2\sigma,t}|$, respectively. For each day, we compare AE of AR-GARCH and QAR-Beta-t-EGARCH, by using $d_{y,t} = AE_{1y,t} - AE_{2y,t}$ (for return forecasting) and $d_{\sigma,t} = AE_{1\sigma,t} - AE_{2\sigma,t}$ (for volatility forecasting). We test whether the mean AE is significantly different from zero, by using the linear regression models $d_{y,t} = c + \epsilon_t$ and $d_{\sigma,t} = c + \epsilon_t$, both estimated by OLS–HAC. A significant and positive c indicates that the predictive performance of QAR-Beta-t-EGARCH is superior to that of AR-GARCH. A significant and negative c indicates that the predictive performance of AR-GARCH is superior to that of QAR-Beta-t-EGARCH.

We present the estimates of c in Table 2. For (D1), the volatility prediction of AR-GARCH dominates for all extreme value definitions, and we also find that the return prediction of AR-GARCH dominates for $k = 6$. According to this result, if tomorrow there would be an extreme value, then AR-GARCH would be used for forecasting. For (D2), the return prediction of QAR-Beta-t-EGARCH dominates for all extreme value definitions, and we also find that the volatility prediction of QAR-Beta-t-EGARCH dominates for $k = 3, 4, 5$. According to this result, if there was an extreme value today, then QAR-Beta-t-EGARCH would be used for forecasting.

[APPROXIMATE LOCATION OF TABLE 2]

Acknowledgments

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Table 1. Descriptive statistics, parameter estimates and model diagnostics

Descriptive statistics			AR-GARCH	QAR-Beta-t-EGARCH
Start date	26th May 1896	c	$0.0002^{***}(0.0000)$	$0.0000^{***}(0.0000)$
End date	10th March 2017	ϕ_1	$0.0837^{***}(0.0066)$	$-0.4427^{***}(0.0986)$
Sample size T	$32,865$	ϕ_2	$-0.0111^{*}(0.0062)$	$-0.2238^{*}(0.1265)$
Minimum	-0.2682	ϕ_3	$0.0152^{**}(0.0062)$	$0.0824(0.1084)$
Maximum	0.1427	ϕ_4	$0.0187^{***}(0.0062)$	$0.3417^{***}(0.0740)$
Mean	0.0002	ϕ_5	$0.0127^{**}(0.0062)$	$0.4226^{***}(0.0993)$
SD	0.0110	ϕ_6	$-0.0030^{*}(0.0065)$	$0.2608(0.1547)$
Skewness	-0.9622	ϕ_7	$-0.0096^{*}(0.0060)$	$0.2682^{**}(0.1183)$
Excess kurtosis	30.6428	ϕ_8	$0.0117(0.0061)$	$0.0714(0.0710)$
		ϕ_{10}	$0.0124^{**}(0.0060)$	$-0.0216^{*}(0.0658)$
		ϕ_{12}	$0.0082(0.0059)$	$0.0278(0.0614)$
		ϕ_{16}	$0.0068(0.0057)$	$0.0661(0.0794)$
		ϕ_{20}	$0.0060(0.0059)$	$0.0106(0.0665)$
		ϕ_{26}	$-0.0088^{*}(0.0056)$	$-0.0775^{*}(0.0687)$
		ϕ_{27}	$0.0065(0.0056)$	$-0.1201^{*}(0.0870)$
		ϕ_{29}	$0.0147^{**}(0.0059)$	$0.1518^{**}(0.0690)$
		ϕ_{30}	$0.0026(0.0060)$	$0.0508(0.0520)$
		θ	NA	$0.1034^{***}(0.0089)$
		ω	$0.0000^{***}(0.0000)$	$-0.0636^{***}(0.0072)$
		α	$0.0353^{***}(0.0041)$	$0.0415^{***}(0.0022)$
		α^{*}	$0.1088^{***}(0.0139)$	$0.0225^{***}(0.0014)$
		β	$0.8944^{***}(0.0090)$	$0.9873^{***}(0.0015)$
		λ_0	$0.0003(0.0002)$	$-4.1309^{***}(0.3957)$
		ν	NA	$6.3818^{***}(0.2337)$
		LL	3.3218	3.3522
		AIC	-6.6422	-6.7029
		BIC	-6.6366	-6.6967
		HQC	-6.6404	-6.7009
		GCLT_λ	0.9262	0.8707
		LB statistic	17.3956	15.1964
		LB p-value	0.9675	0.9886

Notes: Not available (NA). Robust standard errors are in parentheses.
*, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

Table 2. Predictive performance

(D1) $\|y_t - \mu\| \geq 3\sigma$:		
Expected return	4.5137E-06(2.2121E-04)	AR-GARCH and QAR-Beta-t-EGARCH are identical
Volatility	$-$1.4557E-03***(2.2121E-04)	AR-GARCH is superior to QAR-Beta-t-EGARCH
(D1) $\|y_t - \mu\| \geq 4\sigma$:		
Expected return	$-$2.2605E-04(1.5293E-04)	AR-GARCH and QAR-Beta-t-EGARCH are identical
Volatility	$-$2.5490E-03***(4.4906E-04)	AR-GARCH is superior to QAR-Beta-t-EGARCH
(D1) $\|y_t - \mu\| \geq 5\sigma$:		
Expected return	$-$4.0676E-04(2.7928E-04)	AR-GARCH and QAR-Beta-t-EGARCH are identical
Volatility	$-$4.0282E-03***(9.5615E-04)	AR-GARCH is superior to QAR-Beta-t-EGARCH
(D1) $\|y_t - \mu\| \geq 6\sigma$:		
Expected return	$-$8.4623E-04**(3.5905E-04)	AR-GARCH is superior to QAR-Beta-t-EGARCH
Volatility	$-$6.0126E-03***(1.9150E-03)	AR-GARCH is superior to QAR-Beta-t-EGARCH
(D2) $\|y_t - \mu\| \geq 3\sigma$:		
Expected return	5.9304E-04***(1.1380E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	1.3087E-03***(3.5999E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
(D2) $\|y_t - \mu\| \geq 4\sigma$:		
Expected return	8.2777E-04***(2.3840E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	1.7352E-03***(5.6340E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
(D2) $\|y_t - \mu\| \geq 5\sigma$:		
Expected return	1.4739E-03***(5.5474E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	1.8081E-03*(1.0236E-03)	QAR-Beta-t-EGARCH is superior to AR-GARCH
(D2) $\|y_t - \mu\| \geq 6\sigma$:		
Expected return	2.3238E-03**(8.8787E-04)	QAR-Beta-t-EGARCH is superior to AR-GARCH
Volatility	3.5994E-03(2.1960E-03)	AR-GARCH and QAR-Beta-t-EGARCH are identical

Notes: Robust standard errors are in parentheses.

*, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively.

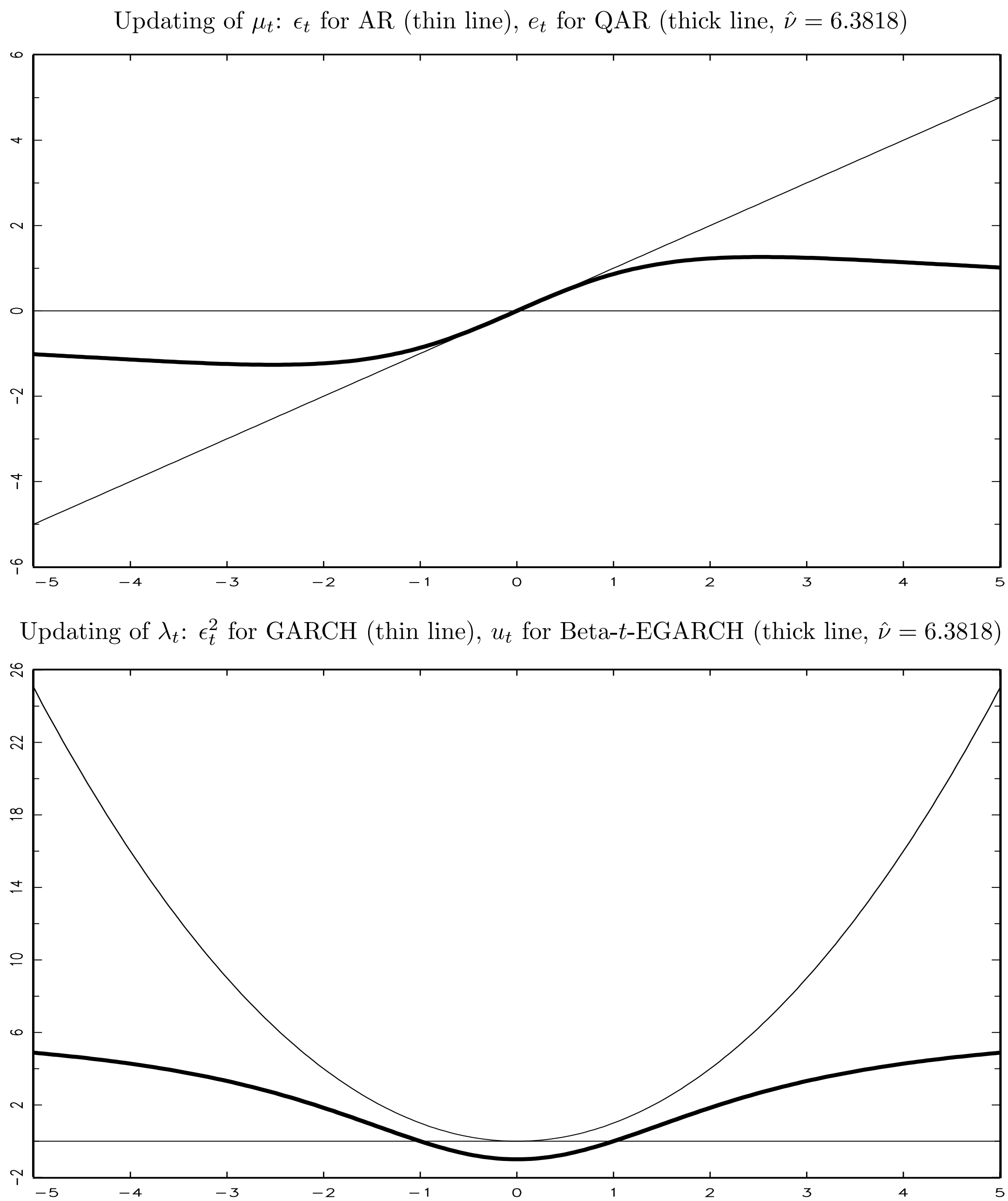

Fig. 1. Updating of μ_t and λ_t after news, as a function of ϵ_t.