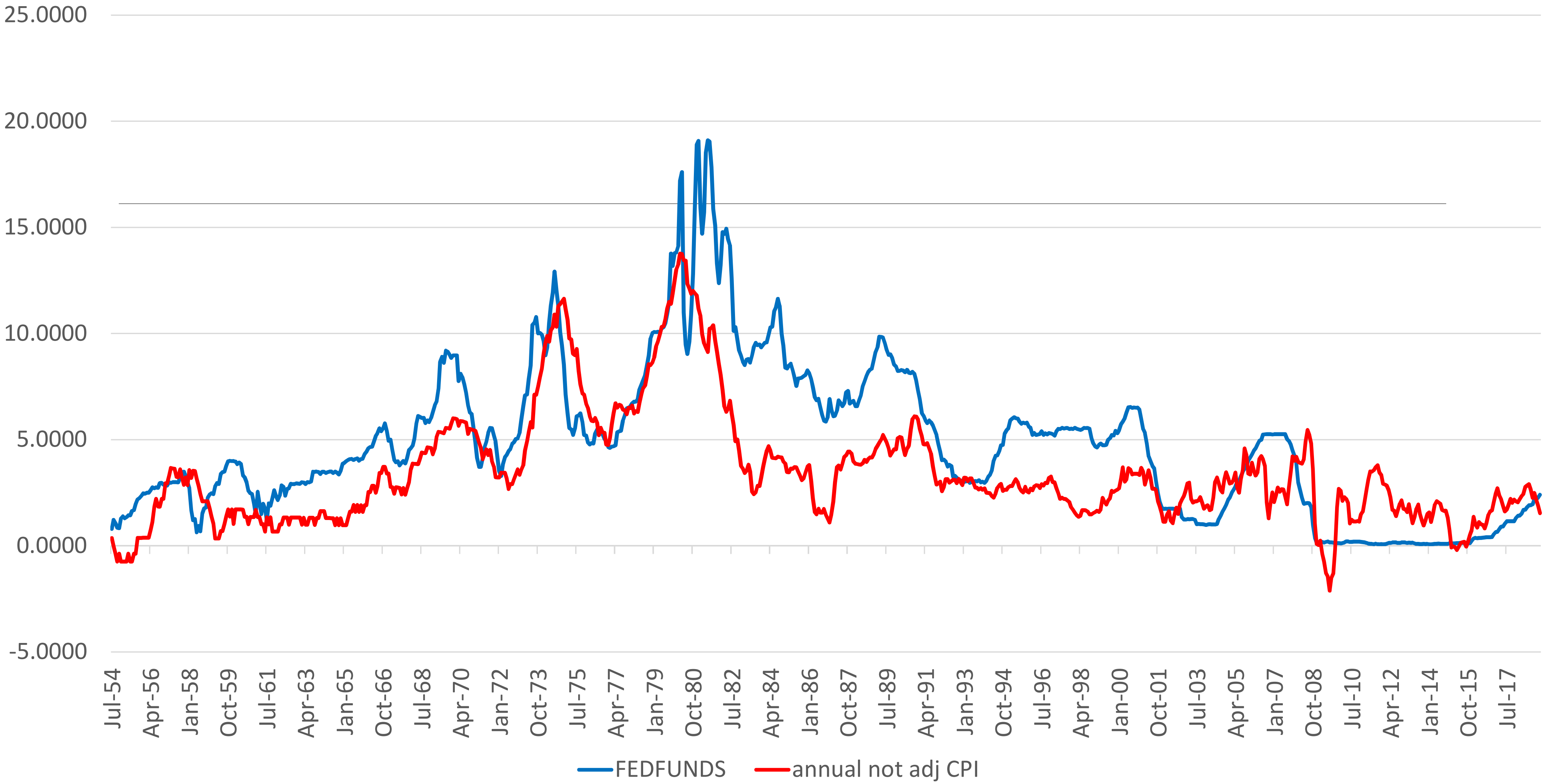
Co-integration and common trends analysis with score-driven models: an application to U.S. macroeconomic data

SZABOLCS BLAZSEK, ALVARO ESCRIBANO, ADRIAN LICHT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FEDFUNDS
annual not adj CPI

This figure shows the evolution of the **Federal Funds Effective Rate** and the **US inflation rate**.

The figure suggests that these two variables move together.

The reason for this co-movement is the fact that there is an underlying **common stochastic trend**, which influences both variables.

The existence of this common stochastic trend is equivalent with co-movement in the time series variables. When there is a common trend underlying to time series, then the time series variables are **co-integrated**.

- **Granger, 1981**. Some Properties of Time Series Data and Their Use in Econometric Model Specification. *Journal of Econometrics* 16.
- **Engle and Granger, 1987**. Co-integration and Error Correction: Representation, Estimation, and Testing. *Econometrica* 55.

Federal Funds Rate is the interest rate at which depository institutions lend reserve balances to other depository institutions overnight.

Federal Funds Rate is the interest rate that the borrowing bank pays to the lending bank to borrow the funds. This is negotiated between the two banks, and the weighted average of this rate across all such transactions is the ***Federal Funds Effective Rate***.

The **Federal Funds Target Rate** is determined by a meeting of the members of the Federal Open Market Committee which normally occurs eight times a year.

The Federal Reserve uses *open market operations* to make the **Federal Funds Effective Rate** follow the **Federal Funds Target Rate**.

Federal Funds Target Rate is a policy rate in the U.S.

With respect to monetary policy guidelines:

Taylor's rule is a proposed guideline for how central banks, such as the Federal Reserve, should alter interest rates in response to changes in economic conditions.

See: Taylor, John B. (1993). "Discretion versus Policy Rules in Practice" *Carnegie-Rochester Conference Series on Public Policy*, vol. 39.

Although there are critiques of this model, it is very popular among central banks to use it in order to forecast the consequences of policy rate changes.

Taylor's rule approximates the responsiveness of the nominal interest rate, as set by the central bank (i.e. Federal Funds Rate), to changes in inflation, output, or other economic conditions.

Taylor's rule recommends that the **Federal Reserve should raise interest rates when inflation is high**.

In particular, the rule describes how, for each 1% increase in inflation, the central bank tends to raise the nominal interest rate by more than 1%.

This aspect of the rule is often called the **Taylor principle**.

Time series data are used from the Federal Funds Effective Rate and the US inflation rate variables:

	Federal funds rate $y_{1,t}$	US inflation rate $y_{2,t}$
Start date	July 1954	July 1954
End date	January 2019	January 2019
Sample size	775	775
Minimum	0.0700	−2.1195
Maximum	19.1000	13.7642
Mean	4.8105	3.4612
Standard deviation	3.6044	2.6763
Skewness	1.0360	1.4888
Excess kurtosis	1.5328	2.4955
Engle–Granger cointegration test:		
ADF-GLS test with constant on y_t	−1.4645(0.1340)	−1.3683(0.1593)
ADF-GLS test with constant on residuals	−2.4570**(0.0136)	
Nyblom–Harvey common trends test:		
Lag-order $m = 50$	0.5841	
Lag-order $m = 100$ — H_0: there is a common trend	0.3709 (p-values)	
Lag-order $m = 200$	0.2991	
Johansen cointegration test (maximum eigenvalue):		
H_0: rank(Π)=0; H_1: rank(Π)=1	17.926**(0.0212)	
H_0: rank(Π)=1; H_1: rank(Π)=2	5.7819(0.2153)	
Lucas outlier-robust co-integration test (t-distribution):		
H_0: rank(Π)=1; H_1: rank(Π)=2	Test statistic, supports H_0: 0.0354	

In the table of data description, the following statistical tests of co-integration/common trend are presented:

- Engle-Granger test of co-integration (1987); *Nobel Prize of 2003*.
- Nyblom-Harvey test of common trends (2000)
- Johansen's test of co-integration (1988, 1991, 1995)
- Outlier robust test of co-integration of Lucas (1997)

Those tests suggest that Federal Funds Effective Rate and US inflation rate are **co-integrated**.

The econometric models of this paper are formulated under the assumption that Federal Funds Effective Rate and US inflation rate are **co-integrated**.

For a policy-maker, it is useful to have forecasts about the effects of current changes in the Federal Funds Effective Rate on the future values of the US inflation rate.

In the literature, several econometric approaches are used for the estimation of relationships among macroeconomic variables.

(1) **Vector autoregressive (VAR) models**.

(2) **Dynamic stochastic general equilibrium (DSGE) models.**

VAR models are very popular classical models. They are easy to estimate, the statistical properties of the estimates are well-studied, and the estimators of dynamic interaction effects are available for practical use.

DSGE models are more recent macroeconomic models with micro foundations, which are becoming popular among central banks. These models are more difficult to estimate and some of the statistical properties of the estimators of DSGE models are currently being developed.

In this paper, an alternative is proposed to these models, which is robust to extreme observations in the macroeconomic variables: ***t*-QVARMA**, which is a **dynamic conditional score (DCS)** model (Harvey, 2013; Creal et al., 2011).

Robustness to extreme observations is relevant, because the statistical inference of VAR and DSGE may be sensitive to extreme observations (for example, during periods of economic or financial crisis).

t-QVARMA(p, q, k) model:

$$y_t = c^* + \mu_t + v_t = c^* + \mu_t^* + \mu_t^\dagger + v_t$$

$$\mu_t^* = \sum_{i=1}^{p} \Phi_i^* \mu_{t-i}^* + \sum_{j=1}^{q} \Psi_j^* u_{t-j}$$

$$\mu_t^\dagger = \mu_{t-1}^\dagger + \sum_{i=1}^{k} \Psi_i^\dagger u_{t-i}$$

$$v_t \sim t_K(0_{K\times 1}, \Sigma, \nu) = t_K[0, \Omega^{-1}(\Omega^{-1})', \nu] \text{ i.i.d.}$$

t-QVARMA model:

The $(K \times 1)$ vector of macroeconomic variables y_t is decomposed into:

(1) the $(K \times 1)$ vector of constant parameters: c^*

(2) the $(K \times 1)$ vector of short-run components: μ_t^*

(3) the $(K \times 1)$ vector of long-run components: $\mu_t^\dagger$

(4) the $(K \times 1)$ vector of reduced-form error terms: v_t.

Both short-run and long-run components depend on past values of all the macroeconomic time series variables of the model.

v_t represents the **unexpected changes (i.e. shocks)** in the values of the macroeconomic variables (for example, a sudden change in the policy rate of the central bank).

t-QVARMA model:

The reduced-form error terms $v_t \sim t_K(0_{K\times 1}, \Sigma_{K\times K}, \nu)$ are correlated, which is measured by using parameter matrix Σ.

In order to correctly measure the effects of unexpected shocks in macroeconomic variables, we introduce the structural-form error term:

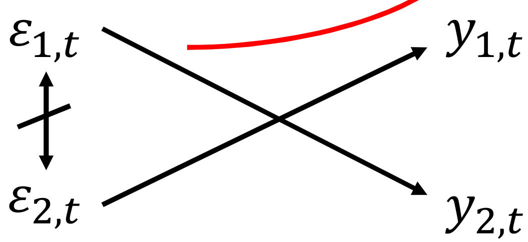
$$\varepsilon_t = \left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega \times v_t$$

Under this definition, $\varepsilon_t \sim t_K(0_{K\times 1}, I_K \times (\nu - 2)/\nu, \nu)$, which are uncorrelated error terms.

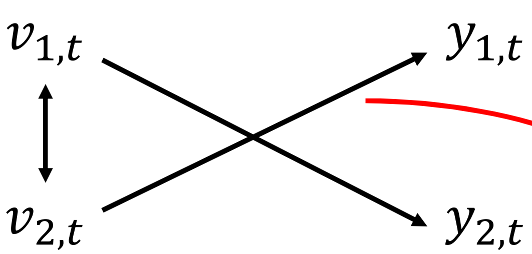
Impulse response analysis: In the analysis of dynamic interaction effects, we evaluate the impact of a one-unit change in each element of ε_t on the future values of all macroeconomic variables.

Motivation for the use of the structural-form errors, instead of the reduced-form errors:

This effects is from $\varepsilon_{1,t}$.



There is no contemporaneous interaction between the structural-form errors.



There is contemporaneous interaction between the reduced-form errors.

It is not clear whether this effects is originated from $v_{2,t}$ or from $v_{1,t}$

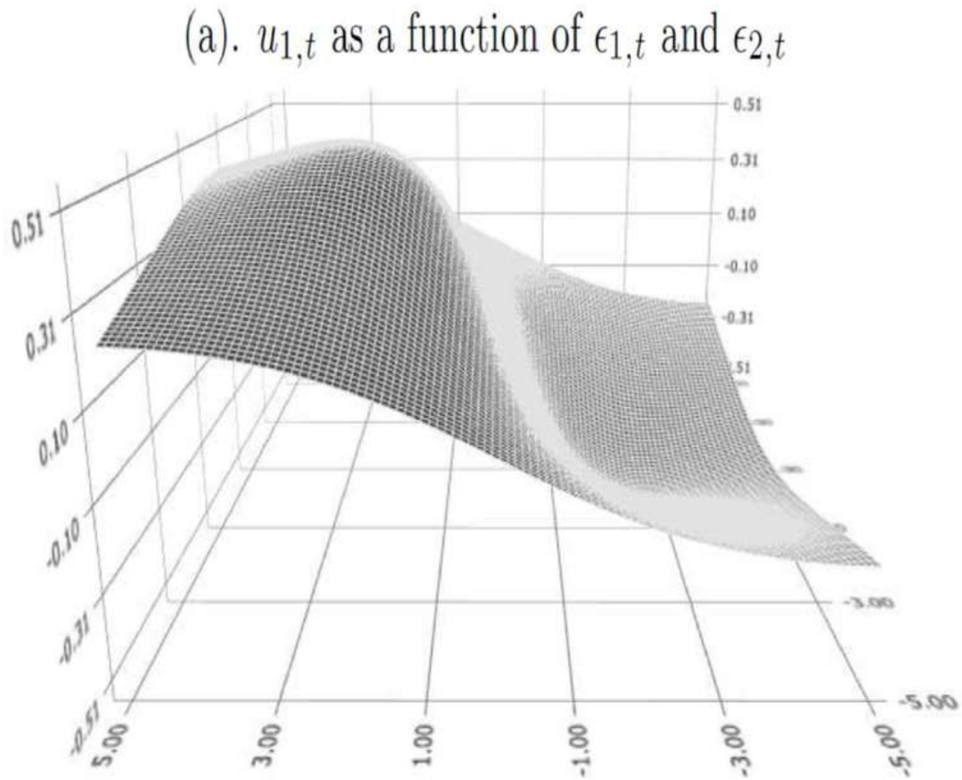
***t*-QVARMA model:**

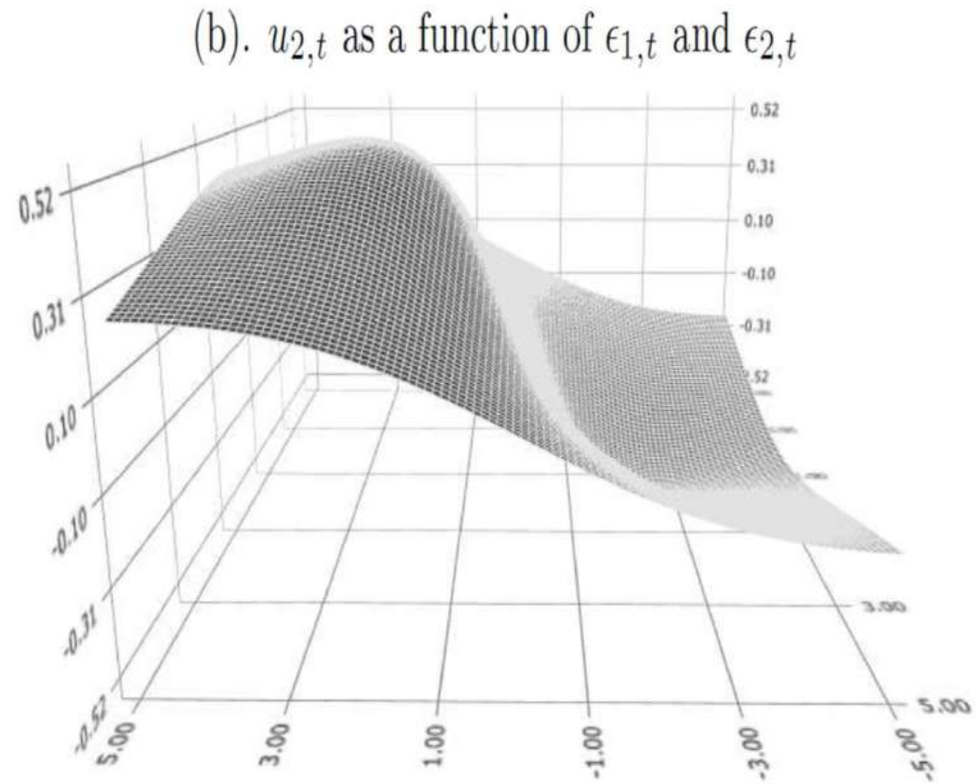
The updating term for both the short-run and the long-run components is the **scaled score function** u_t (Harvey, 2013), which is a nonlinear transformation of the reduced-form error term v_t.

The nonlinear transformation of v_t is such that it **reduces the effects of extreme observations in the macroeconomic variables**, and, by doing this, both the short-run and the long-run components provide more precise measurements of the dynamic interaction effects among the macroeconomic variables of the model.

Note: The sensitivity of the estimation of classical VAR models to extreme observations is one of the motivations of the present paper.

u_t reduces the effects of extreme observations on the macroeconomic variables:





***t*-QVARMA model:**

Why the need for the use of both short-run and long-run effects?

When an unexpected change in a macroeconomic variable occurs, then this may have both short-run and long-run effects on the future values of other macroeconomic variables (see the **Granger representation** of VAR models; Johansen, 1995).

The **short-run effects** may be very significant for the nearest future periods, but those effects disappear over time (i.e. the short-run effects converge to zero in future).

The **long-run effects** may also be very significant, and they do not disappear. Those effects modify the levels of other variables (i.e. the long-run effects converge to a non-zero value in the future).

***t*-QVARMA model:**

By separately measuring the short-run and the long-run effects, the econometric model may provide more precise forecasts of the dynamic interaction effects among the macroeconomic variables of the model.

The decomposition of short-run and long-run effects may also give relevant information for policy-makers.

t-QVARMA model:

Co-integration (i.e. common trend) relationship between the variables is imposed by using a reduced rank specification for parameter matrix $\Psi_i^{\dagger}$.

For two-dimensional t-QVARMA(1,1,1) case, this reduced rank specification is given by:

$$\begin{bmatrix} \mu_{1,t}^{\dagger} \\ \mu_{2,t}^{\dagger} \end{bmatrix} = \begin{bmatrix} \mu_{1,t-1}^{\dagger} \\ \mu_{2,t-1}^{\dagger} \end{bmatrix} + \begin{bmatrix} \Psi_{1,11}^{\dagger} & \Psi_{1,12}^{\dagger} \\ \kappa\Psi_{1,11}^{\dagger} & \kappa\Psi_{1,12}^{\dagger} \end{bmatrix} \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix}$$

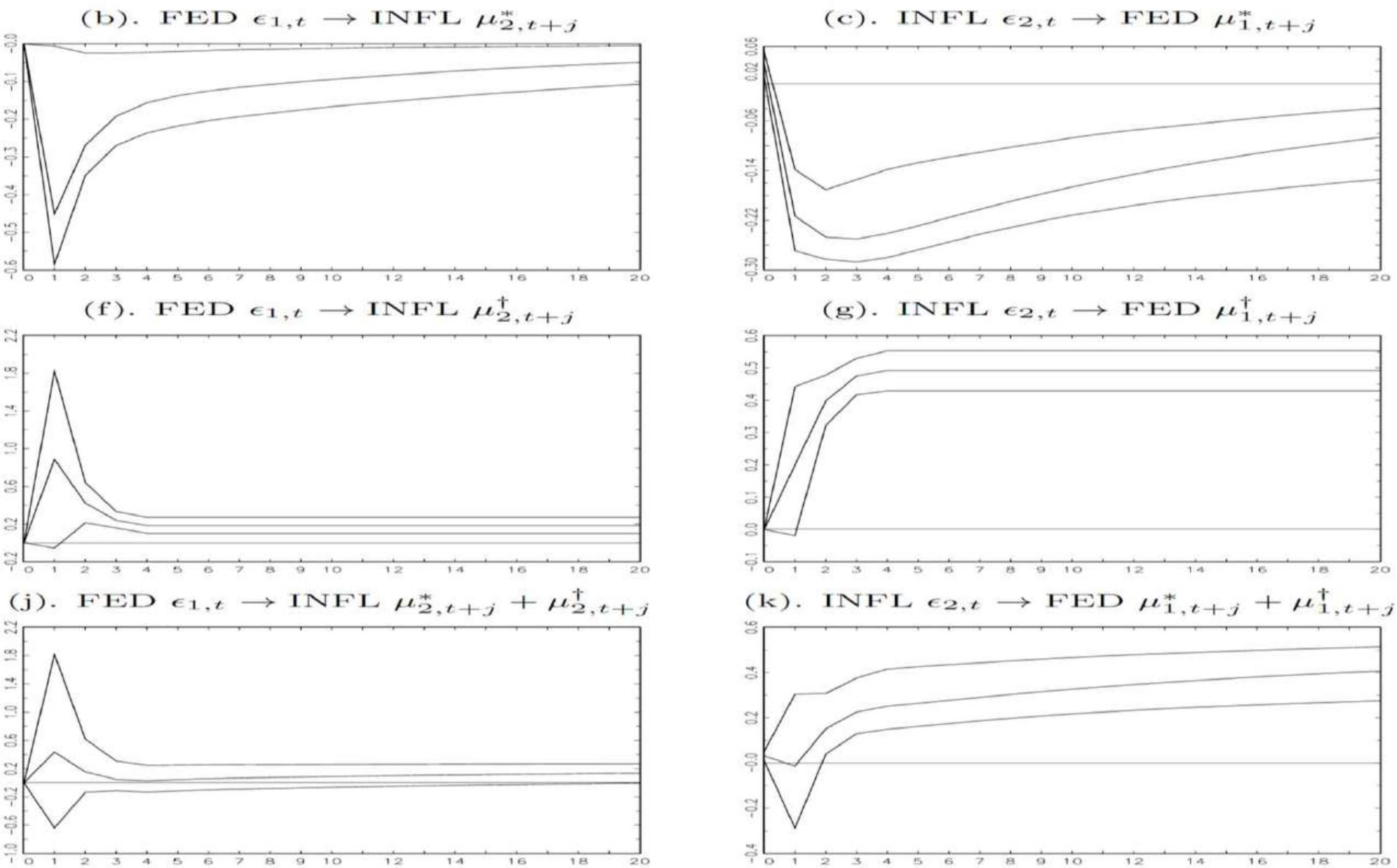
and $\mu_{1,t}^{\dagger}$ defines the common stochastic trend.

t-QVARMA(1,1,4)

Short-run effects:
negative

Long-run effects:
positive

Total effects:
non-significant,
positive after
several leads



A limiting special case of *t*-QVARMA is the Gaussian-QVARMA model:

$$y_t = c^* + \mu_t^* + \mu_t^\dagger + v_t$$

$$\mu_t^* = \sum_{i=1}^{p} \Phi_i^* \mu_{t-i}^* + \sum_{j=1}^{q} \Psi_j^* v_{t-j}$$

$$\mu_t^\dagger = \mu_{t-1}^\dagger + \sum_{i=1}^{k} \Psi_i^\dagger v_{t-i}$$

$$v_t \sim N_K(0_{K\times 1}, \Sigma) = N_K[0, \Omega^{-1}(\Omega^{-1})'] \text{ i.i.d.}$$

We obtain this model for $\nu \rightarrow \infty$ from t-QVARMA.

Gaussian-QVARMA includes classical Gaussian-VARMA specifications for the short-run and the long-run components.

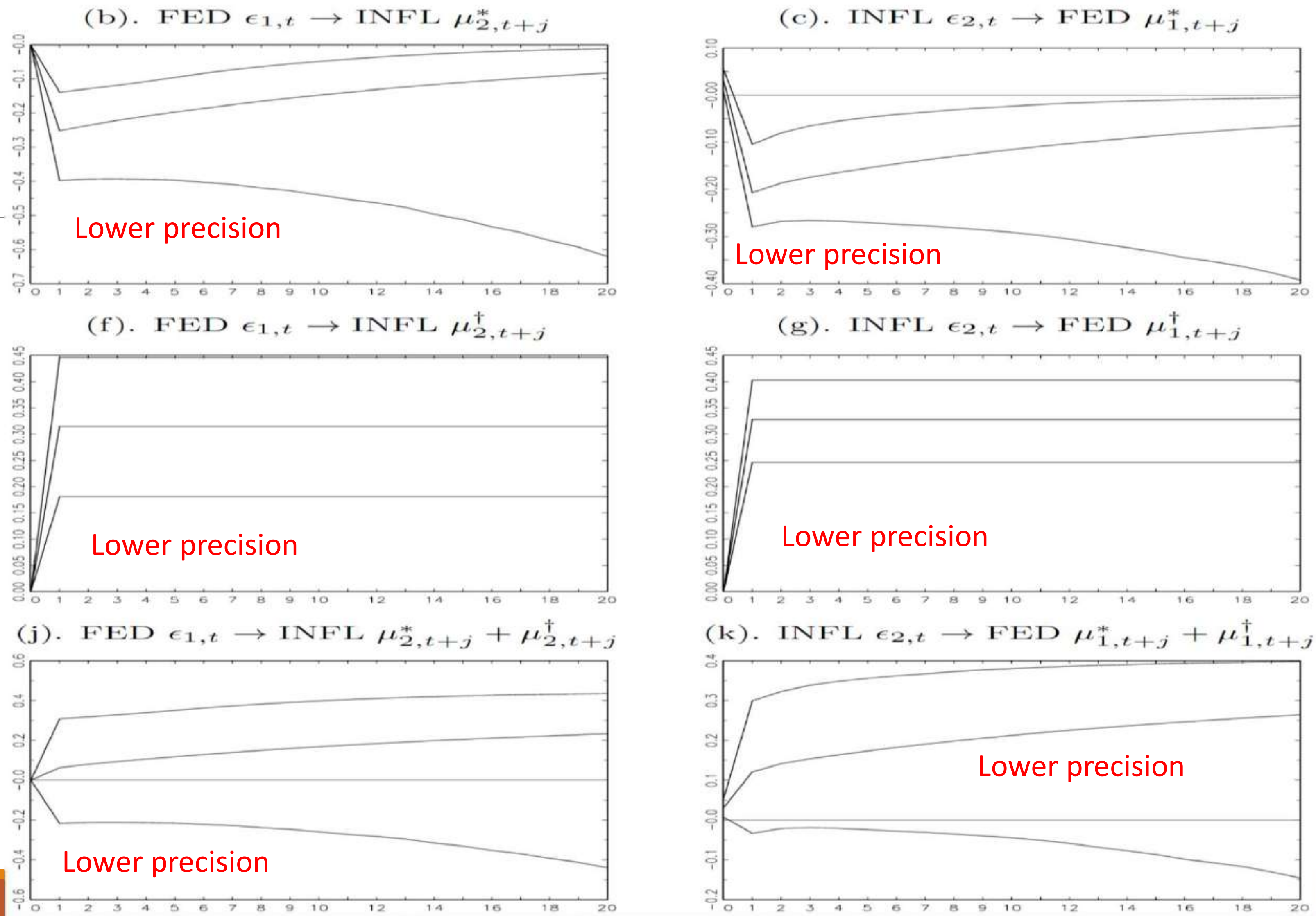
Due to the use of the normal distribution, this model is sensitive to extreme observations.

Gaussian-QVARMA(1,1,1)

Short-run effects:
negative

Long-run effects:
positive

Total effects:
non-significant



As aforementioned, **VAR models** are popular in practice.

Gaussian-VAR(p):

$y_t = c + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + v_t$ where $v_t \sim N_K(0_{K\times 1}, \Sigma)$.

This model is estimated by using the following **Vector Error Correction Model (VECM)** representation:

$$\Delta y_t = \alpha\beta'(y_{t-1} - c) + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + v_t$$

where $v_t \sim N_K(0_{K\times 1}, \Sigma)$.

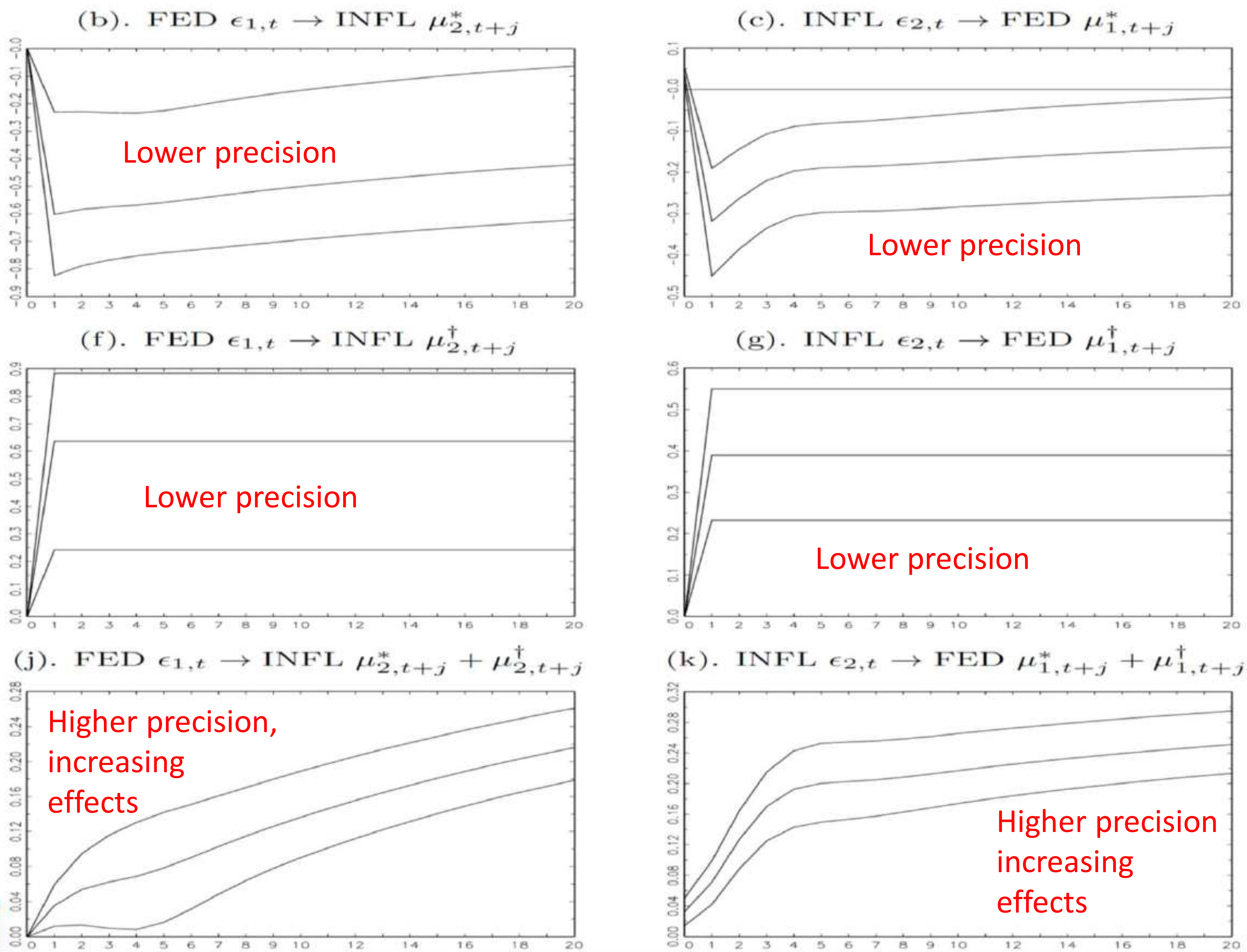
Gaussian-VAR(p) is sensitive to extreme observations in the macroeconomic variables, due to the Gaussian error term v_t.

Gaussian-VAR(1)

Short-run effects:
negative

Long-run effects:
positive

Total effects:
positive

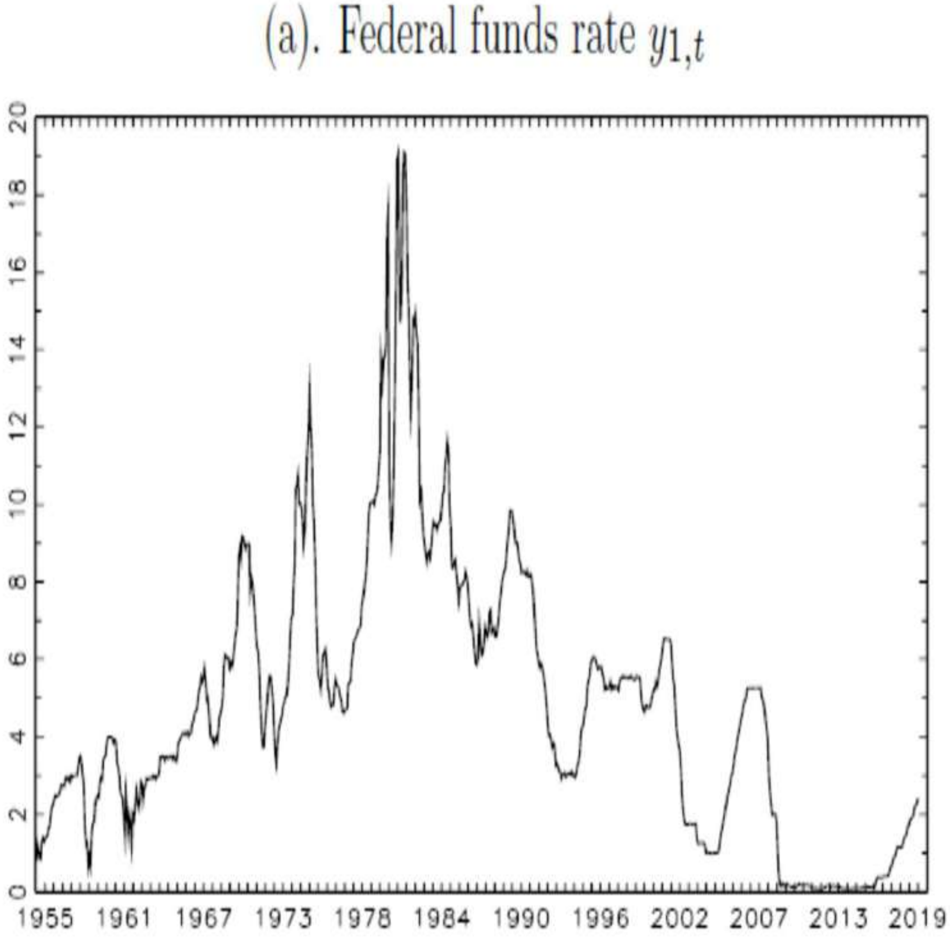


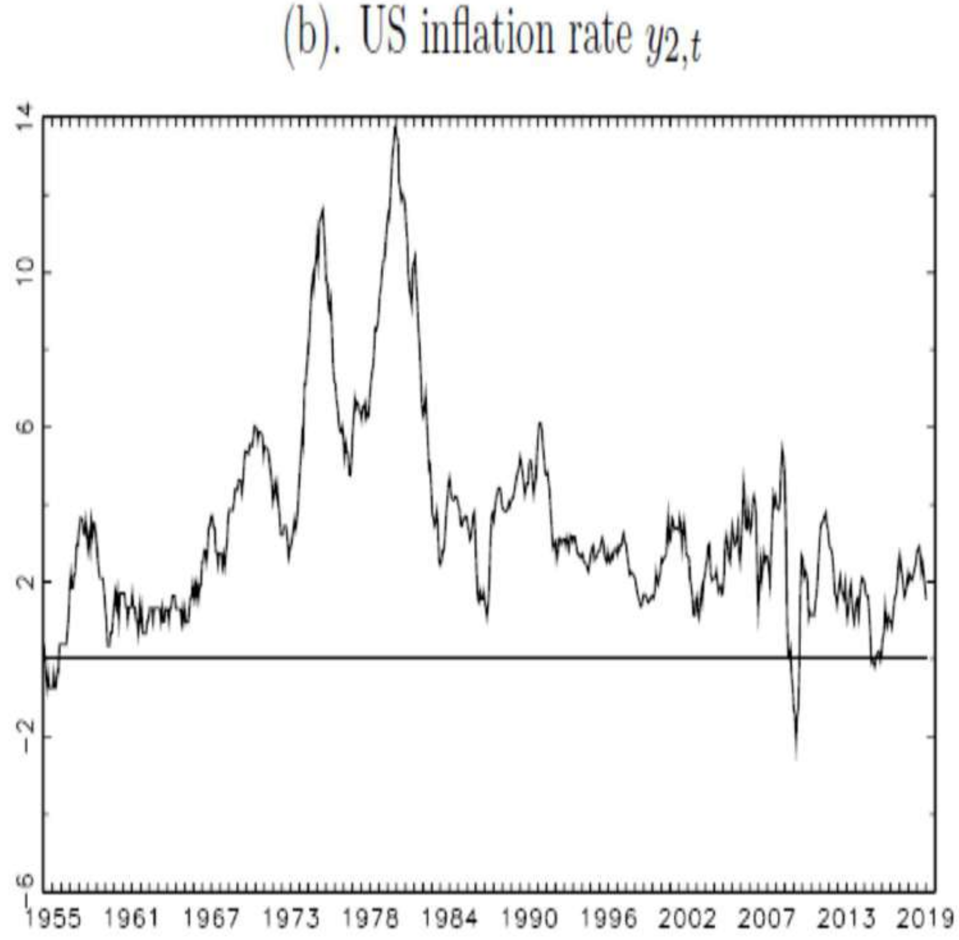
To find out, which of the competing models is supported for the measurement of dynamic interaction effects, the statistical performances of all models are compared:

	t-QVARMA(1,1,1)		t-QVARMA(1,1,4)		Gaussian-QVARMA(1,1,1)			Gaussian-VAR(1)		Gaussian-VAR(4)	
$\Phi^*_{1,11}$	0.5162***	(0.0453)	0.5132***	(0.1540)	0.8407***	(0.1207)	c_1	0.2911	(0.4135)	0.6452*	(0.3295)
$\Phi^*_{1,12}$	-0.1097*	(0.0641)	-0.1285**	(0.0581)	-0.0975	(0.0998)	c_2	0.1255	(0.5412)	0.4489	(0.4368)
$\Phi^*_{1,21}$	-0.3416***	(0.0532)	-0.4033***	(0.0380)	-0.6617**	(0.3202)	α_1	-0.0136+	(0.0083)	-0.0181**	(0.0080)
$\Phi^*_{1,22}$	0.8789***	(0.0501)	0.8141***	(0.0573)	0.3145+	(0.1962)	α_2	0.0162***	(0.0053)	0.0153***	(0.0048)
$\Psi^*_{1,11}$	-0.6608***	(0.1949)	1.3881+	(0.8681)	0.4696***	(0.1540)	β_1	-1.3486***	(0.2812)	-1.3858***	(0.2467)
$\Psi^*_{1,12}$	-0.5907***	(0.1476)	-3.1473+	(1.9832)	-0.8159***	(0.2347)	β_2	1.0000		1.0000	
$\Psi^*_{1,21}$	-1.4040***	(0.1817)	-0.3083	(0.5625)	-0.6148***	(0.1104)	$\Gamma_{1,11}$	NA		0.4395***	(0.0333)
$\Psi^*_{1,22}$	1.2610***	(0.1076)	-0.3061	(1.2477)	0.5840***	(0.1727)	$\Gamma_{1,12}$	NA		0.0796*	(0.0430)
$\Psi^\dagger_{1,11}$	2.7844***	(0.2398)	0.7974	(0.8530)	0.9381***	(0.1567)	$\Gamma_{1,21}$	NA		0.0519**	(0.0239)
$\Psi^\dagger_{1,12}$	0.6299***	(0.1579)	3.0949+	(1.9810)	0.9293***	(0.2334)	$\Gamma_{1,22}$	NA		0.2619***	(0.0331)
$\Psi^\dagger_{2,11}$	NA		1.1755*	(0.6946)	NA		$\Gamma_{2,11}$	NA		-0.1371***	(0.0358)
$\Psi^\dagger_{2,12}$	NA		-1.6167	(1.5833)	NA		$\Gamma_{2,12}$	NA		-0.0398	(0.0446)
$\Psi^\dagger_{3,11}$	NA		0.4578**	(0.1971)	NA		$\Gamma_{2,21}$	NA		0.0526**	(0.0262)
$\Psi^\dagger_{3,12}$	NA		-0.6375+	(0.4086)	NA		$\Gamma_{2,22}$	NA		0.1001***	(0.0335)
$\Psi^\dagger_{4,11}$	NA		0.1070*	(0.0636)	NA		$\Gamma_{3,11}$	NA		-0.0221	(0.0340)
$\Psi^\dagger_{4,12}$	NA		-0.1985**	(0.0900)	NA		$\Gamma_{3,12}$	NA		-0.0180	(0.0428)
κ	0.5805***	(0.0504)	0.6499***	(0.0391)	0.7199***	(0.0527)	$\Gamma_{3,21}$	NA		0.0201	(0.0248)
ν	9.8545***	(0.2913)	9.1904***	(0.5548)	NA		$\Gamma_{3,22}$	NA		-0.0819**	(0.0332)
LL	-0.9362		**-0.8954**		-0.9684		LL	-1.1207		-0.9638	
AIC	1.9137		**1.8552**		1.9756		AIC	2.2620		1.9793	
BIC	2.0097		**2.0053**		2.0656		BIC	2.3101		2.0995	
HQC	1.9506		**1.9130**		2.0102		HQC	2.2805		2.0255	

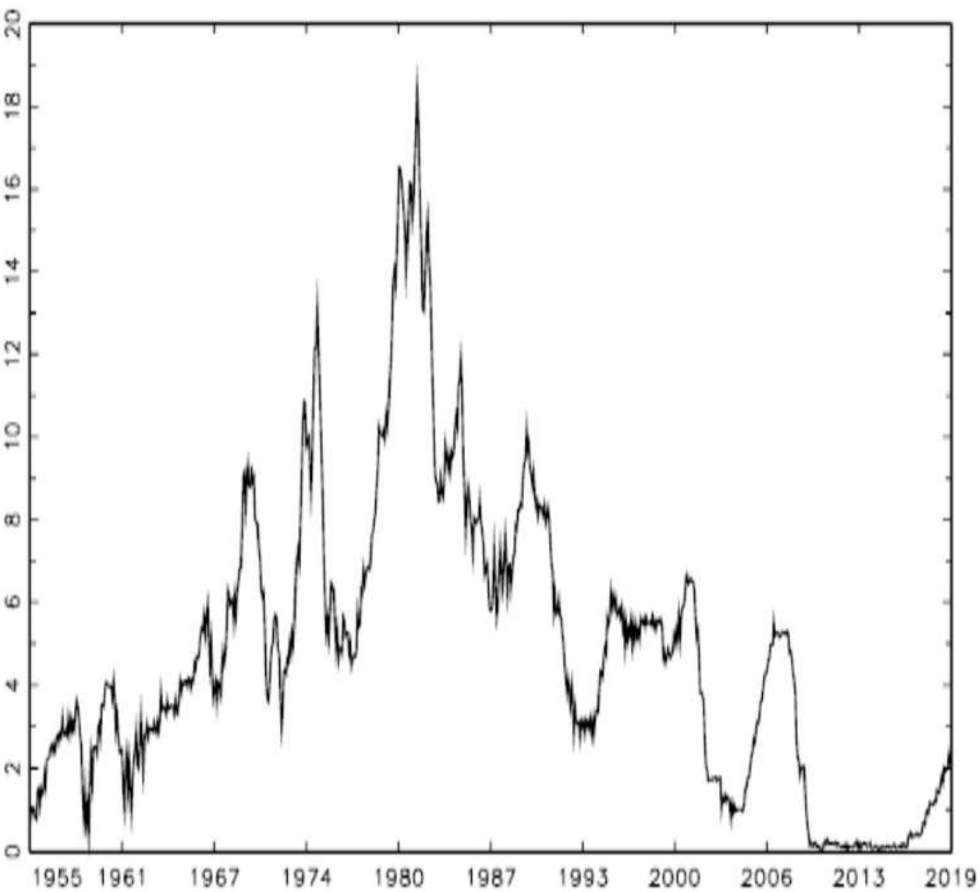
The likelihood-based model selection metrics support the use of the *t*-QVARMA(1,1,4) model.

Estimates of Federal Funds Effective Rate and US inflation rate decomposition for the *t*-QVARMA(1,1,4) model:

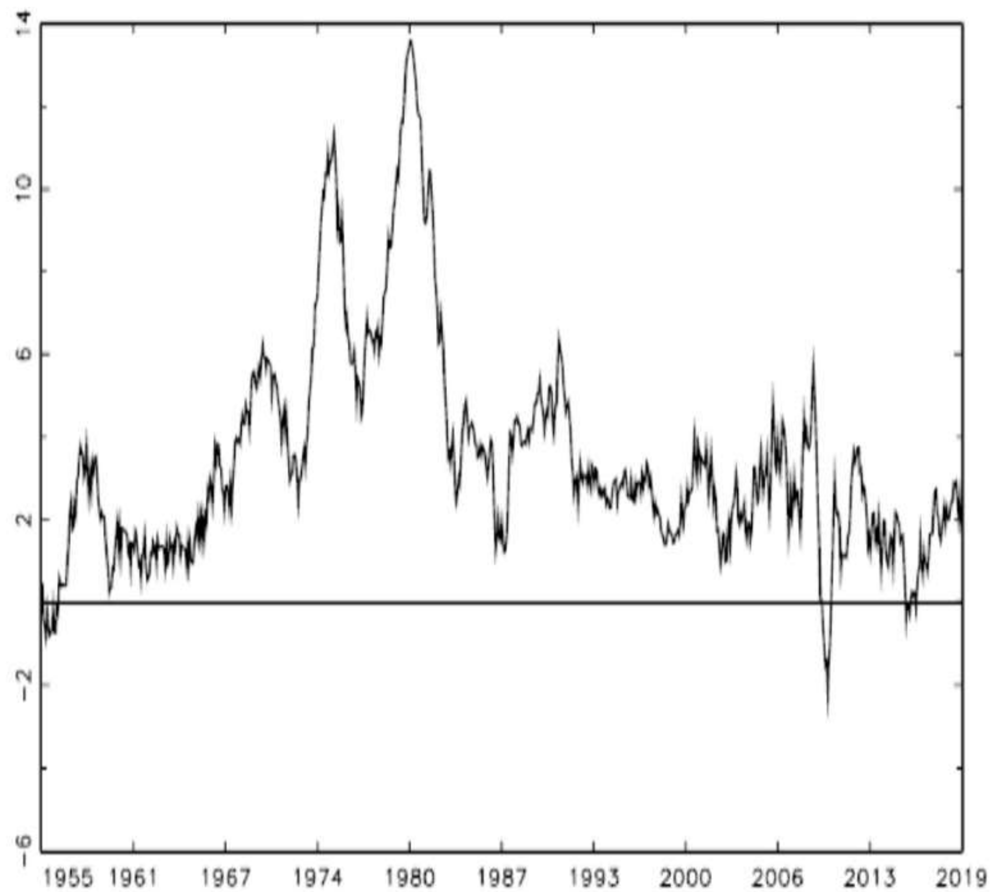

(a). Federal funds rate $y_{1,t}$
0
2
4
6
8
10
12
14
16
18
20
1955 1961 1967 1973 1978 1984 1990 1996 2002 2007 2013 2019

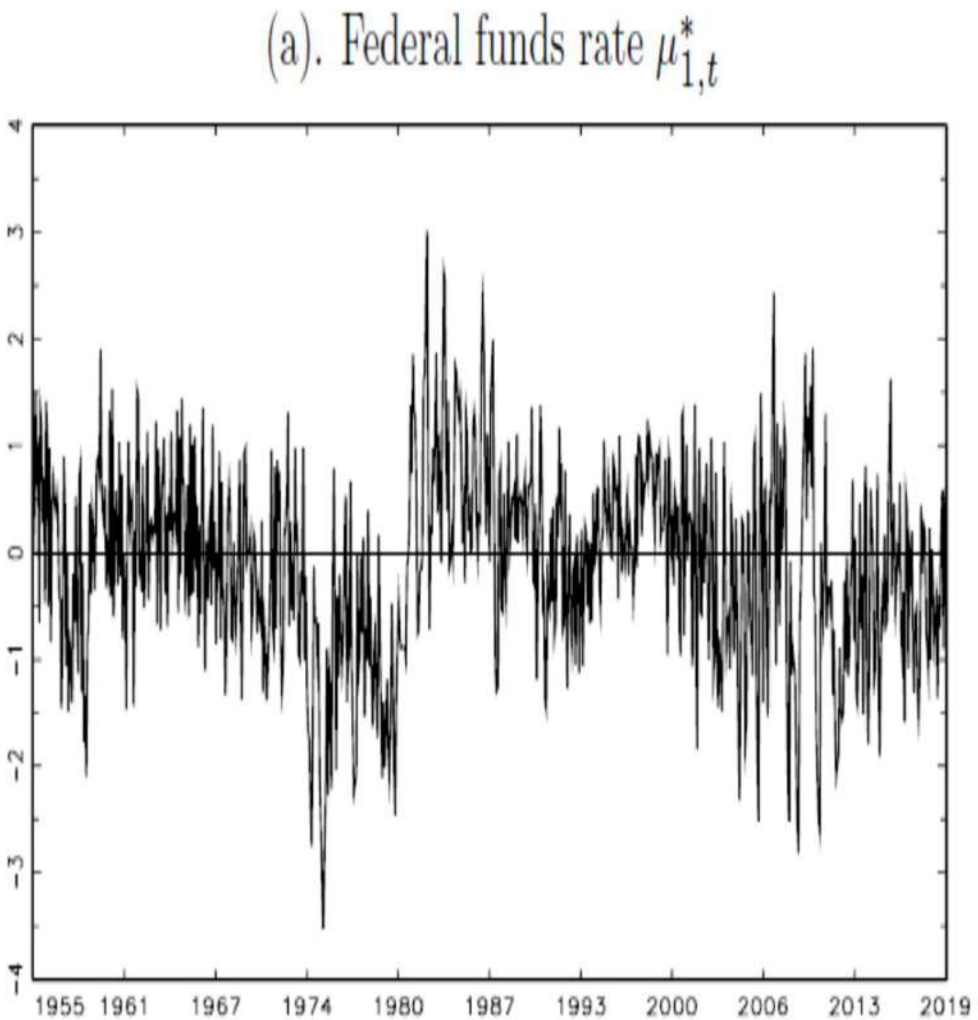

(b). US inflation rate $y_{2,t}$
−6
−2
2
6
10
14
1955 1961 1967 1973 1978 1984 1990 1996 2002 2007 2013 2019

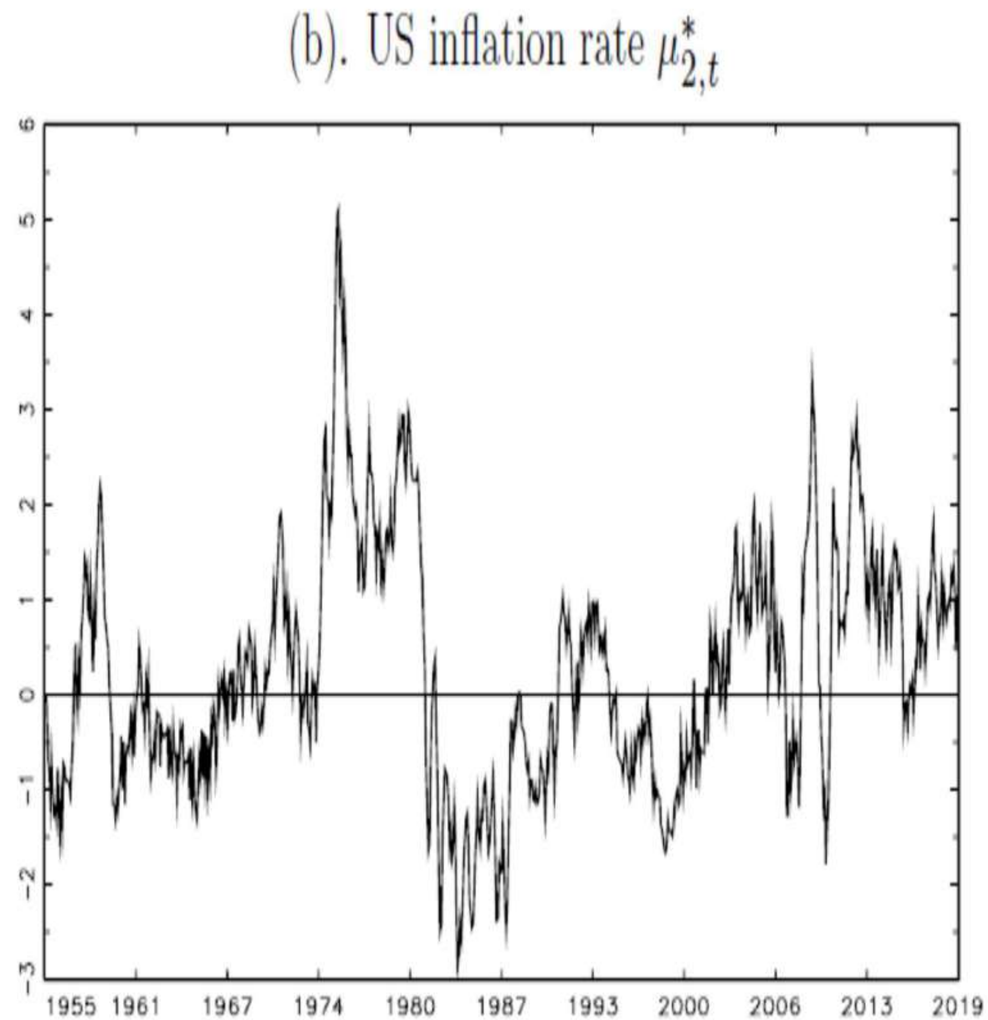
(e). Federal funds rate $\mu^*_{1,t} + \mu^\dagger_{1,t}$



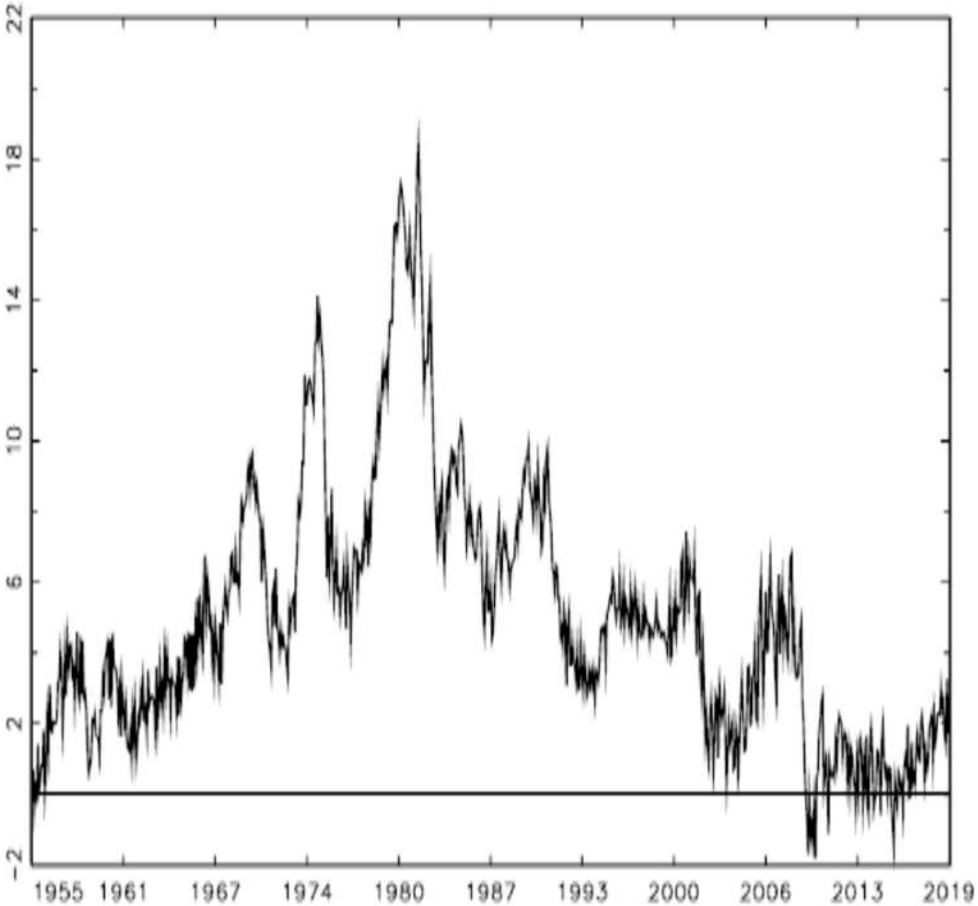
(f). US inflation rate $\mu^*_{2,t} + \mu^\dagger_{2,t}$



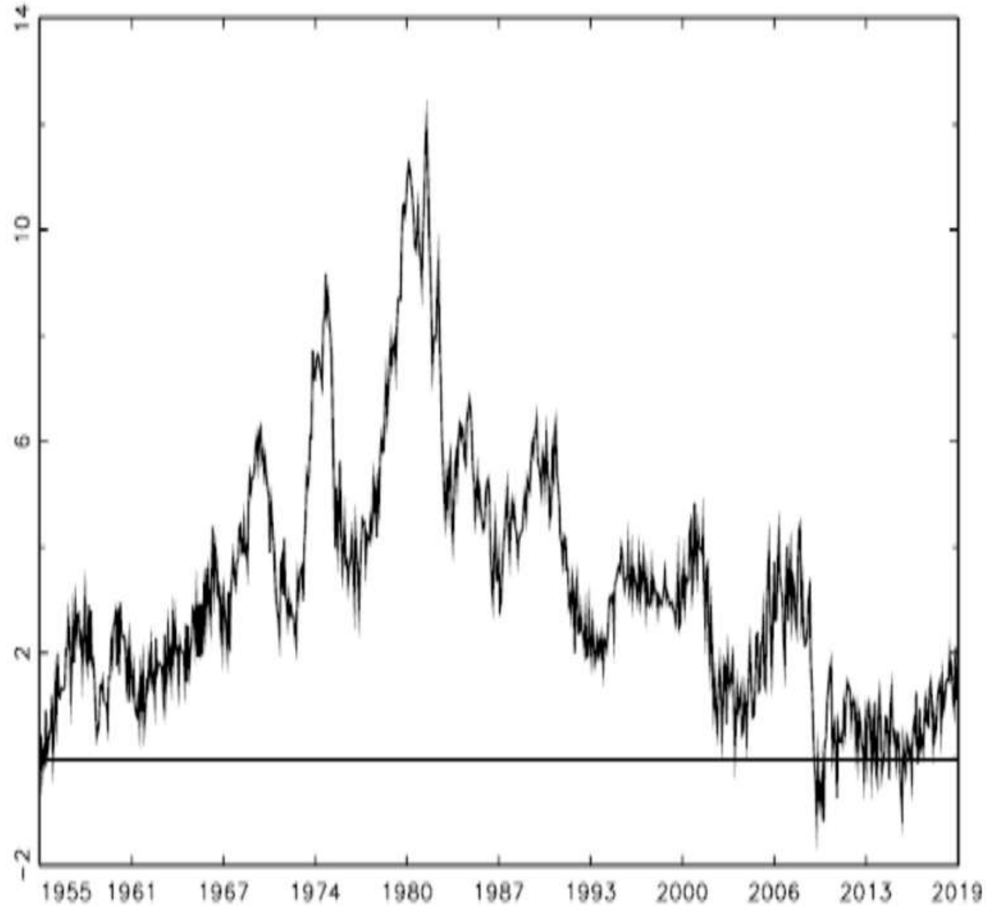

(a). Federal funds rate $\mu^*_{1,t}$
4
3
2
1
0
−1
−2
−3
−4
1955 1961 1967 1974 1980 1987 1993 2000 2006 2013 2019

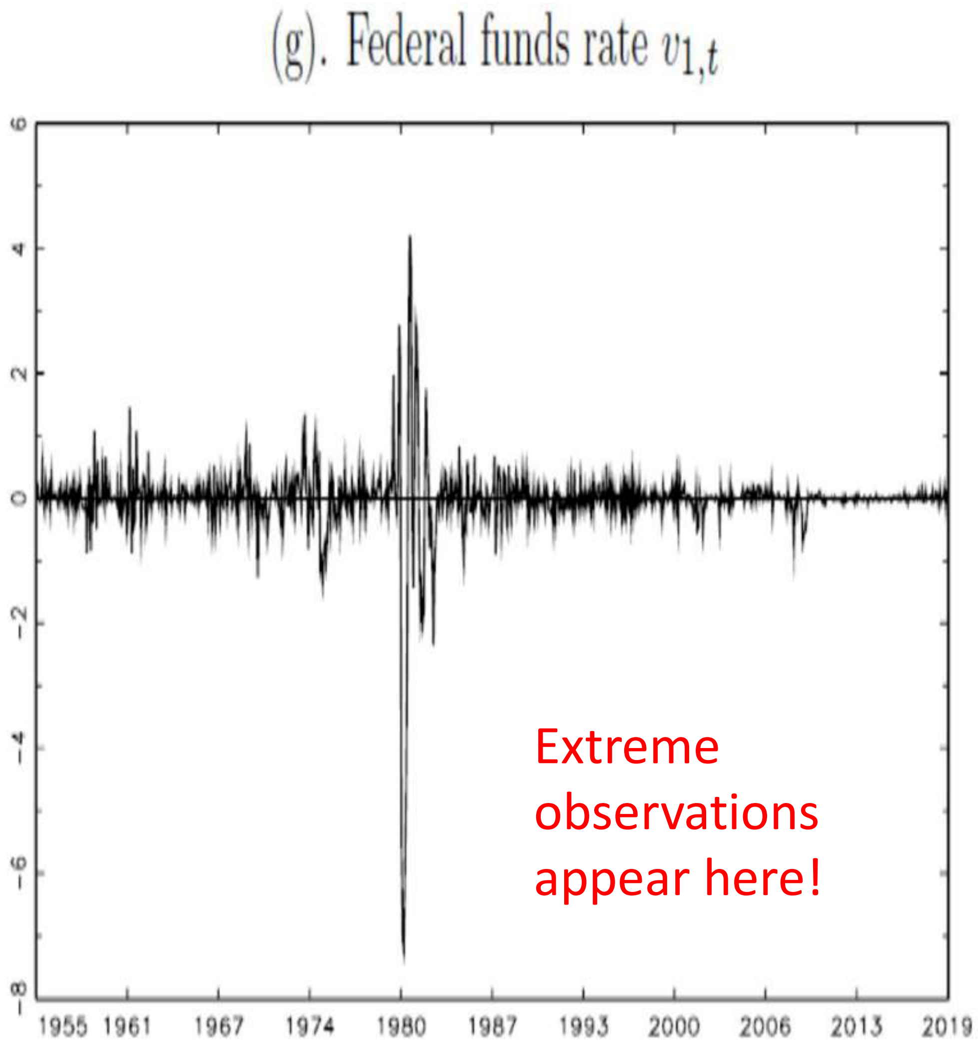

(b). US inflation rate $\mu^*_{2,t}$
6
5
4
3
2
1
0
−1
−2
−3
1955 1961 1967 1974 1980 1987 1993 2000 2006 2013 2019

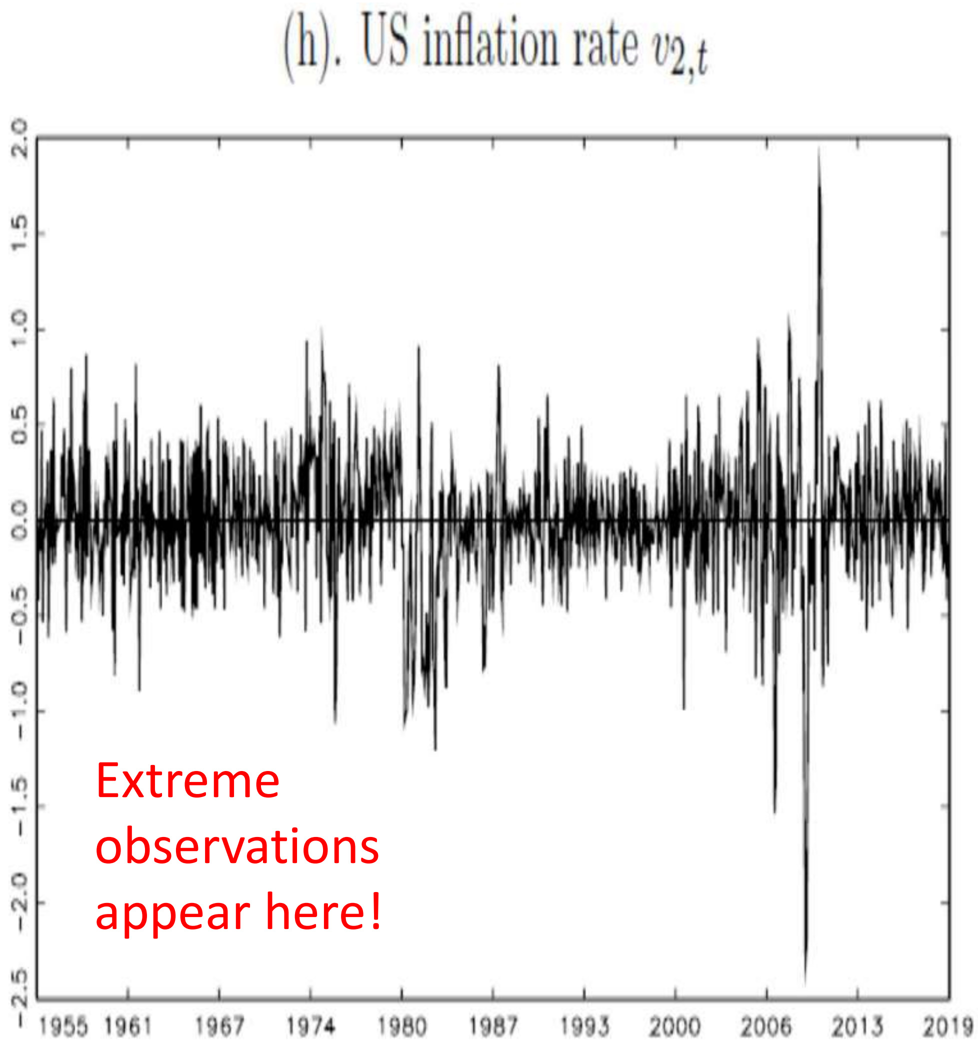
(c). Federal funds rate $\mu^{\dagger}_{1,t}$



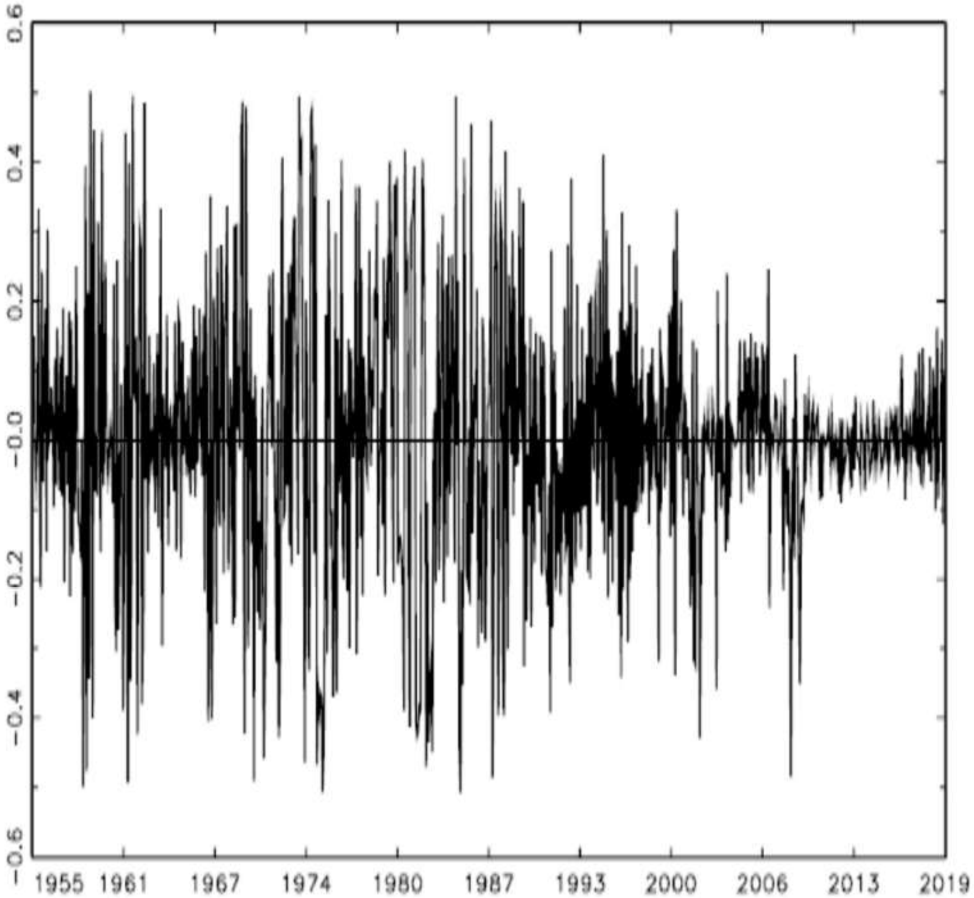
(d). US inflation rate $\mu^{\dagger}_{2,t}$




(g). Federal funds rate $v_{1,t}$
6
4
2
0
−2
−4
−6
−8
1955 1961 1967 1974 1980 1987 1993 2000 2006 2013 2019
Extreme observations appear here!

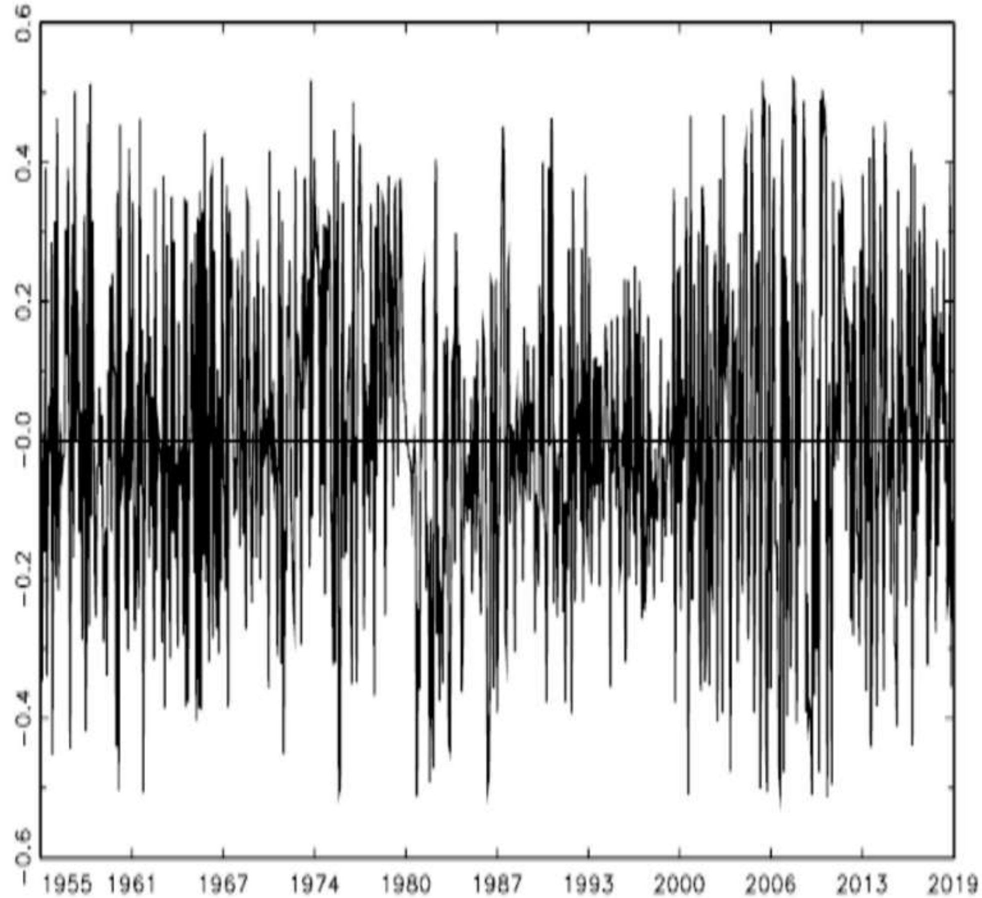

(h). US inflation rate $v_{2,t}$
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
1955 1961 1967 1974 1980 1987 1993 2000 2006 2013 2019
Extreme observations appear here!

(i). Federal funds rate $u_{1,t}$



Protected against extreme observations

(j). US inflation rate $u_{2,t}$



Thank you for your attention.

SBLAZSEK@UFM.EDU

ADRIAN.LICHT@GMAIL.COM