MARKOV-SWITCHING SCORE-DRIVEN MULTIVARIATE MODELS

OUTLIER-ROBUST MEASUREMENT OF THE RELATIONSHIPS BETWEEN WORLD CRUDE OIL PRODUCTION AND US INDUSTRIAL PRODUCTION

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Motivation

1. United States (US) crude oil production has recently surpassed 12 million barrels per day (top crude oil producer country in the world), which is partly due to the improved fracking technologies.
2. Renegotiation of NAFTA → USMCA
3. Unconventional monetary policy after the 2008 US financial crisis.

All these events and technological developments are in relation to the **world crude oil production** and **US industrial production** relation.

Contribution

New **Seasonal-QVAR** and **Markov-switching (MS) Seasonal-QVAR** models.

Seasonal-QVAR is **robust to outliers**; *MS-Seasonal-QVAR* is robust to outliers and it may **identify structural changes**.

The new models are alternatives to classical Gaussian multivariate models:

- **Seasonal-VARMA**
- **Basic structural model**

Literature on oil price shocks

Baumeister and Peersman (2013): (i) global oil production, (ii) real acquisition cost of imported crude oil of US refineries, (iii) US real GDP, and (iv) US consumer prices; 1974:Q1 to 2011:Q1. *Oil supply shocks have negative effects on real GDP*.

Baumeister and Hamilton (2017): world crude oil production, world industrial production, real crude oil price, and world crude oil stock; January 1958 to December 2016. *Oil supply shocks have negative effects on world industrial production.*

Kilian and Lütkepohl (2017): world crude oil production, business cycle index (global real economic activity), and real price of crude oil; February 1973 to December 2017. Oil supply shocks *have non-significant effects* on real economic activity.

Data of this paper: February 1973 to April 2019

Sources for crude oil production: (i) Kilian and Lütkepohl (2017); (ii) Bloomberg (ticker: DWOPWRLD).

Source for US industrial production: OECD.

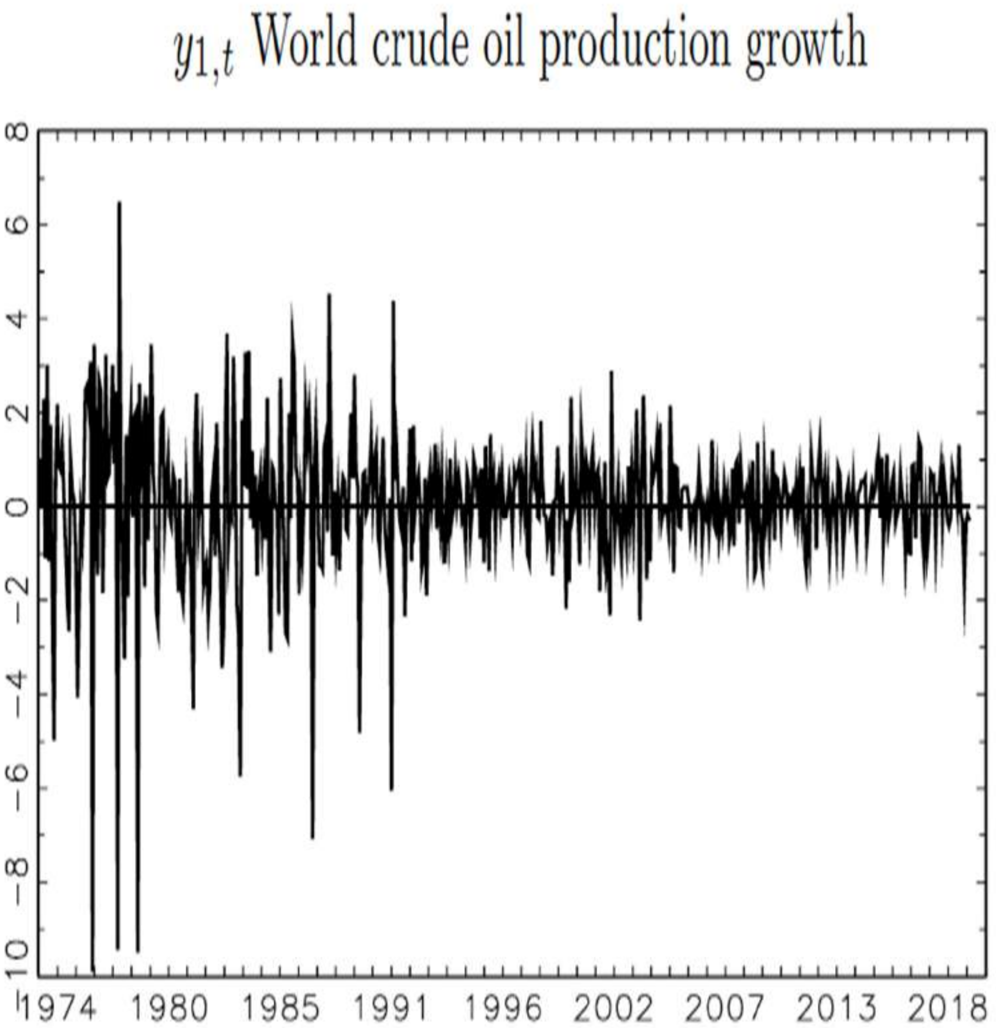
This dataset includes the following variables:

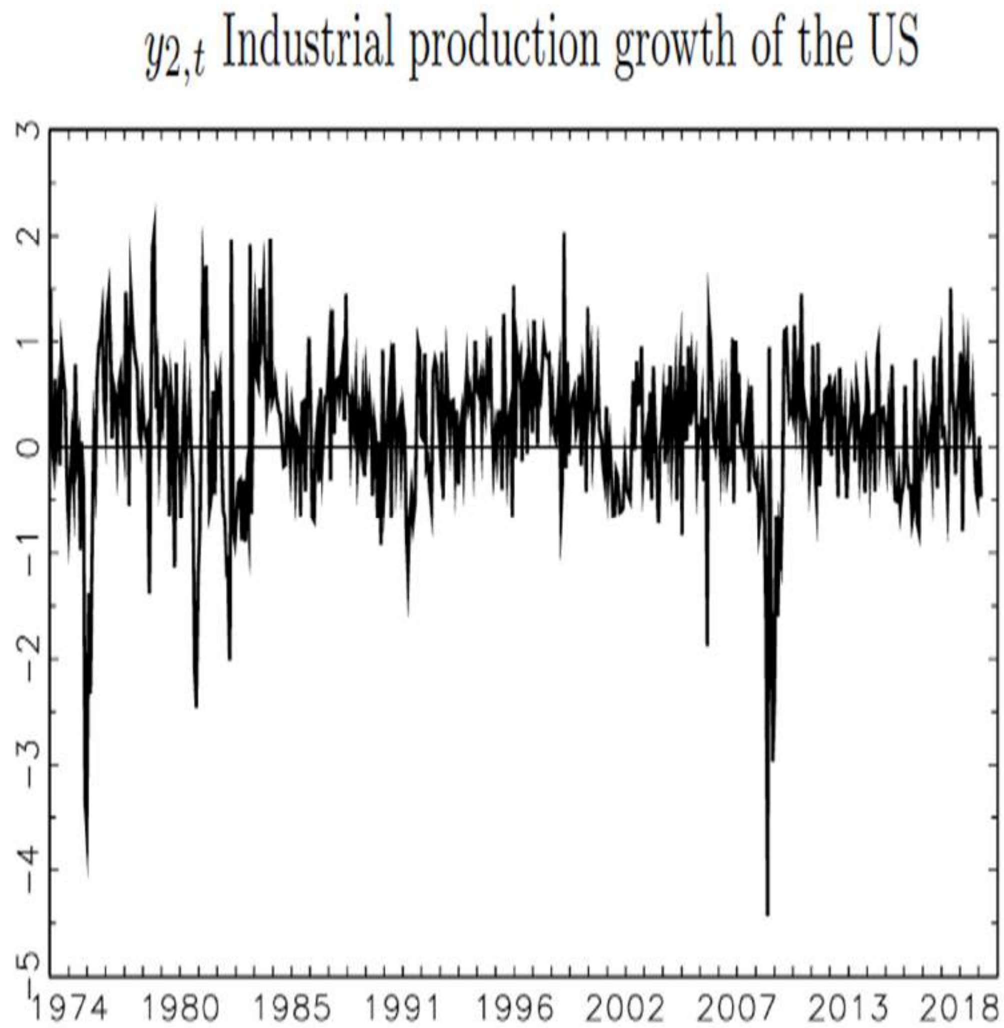
(i) y_{1t} **% change in monthly world crude oil production**.

(ii) y_{2t} **% change in monthly industrial production**.

Panel A. Descriptive statistics	World crude oil production growth y_{1t}	US industrial production growth y_{2t}
Start date	February 1973	February 1973
End date	April 2019	April 2019
Sample size T	555	555
Minimum	−9.9073	−4.4337
Maximum	6.4986	2.0506
Mean	0.0732	0.1585
Standard deviation	1.5224	0.7153
Skewness	−1.6452	−1.3149
Excess kurtosis	10.4297	6.1412
ADF	−25.5561***	−8.3432***

Panel B. Seasonality effects	World crude oil production growth y_{1t}	US industrial production growth y_{2t}
δ_{Jan}	$-0.9737^{***}(0.3301)$	$-0.0448(0.1244)$
δ_{Feb}	$0.2798(0.2327)$	$0.2180^{*}(0.1133)$
δ_{Mar}	$0.0117(0.1485)$	$0.1559^{*}(0.0915)$
δ_{Apr}	$-0.2615^{*}(0.1536)$	$0.1370(0.1008)$
δ_{May}	$-0.1636(0.1731)$	$0.1662^{*}(0.0950)$
δ_{Jun}	$0.1767(0.1951)$	$0.1577^{**}(0.0672)$
δ_{Jul}	$0.7302^{***}(0.1966)$	$0.1652^{**}(0.0758)$
δ_{Aug}	$-0.2276(0.2050)$	$0.2212^{**}(0.0946)$
δ_{Sep}	$0.4115^{*}(0.2467)$	$0.1157(0.1303)$
δ_{Oct}	$0.4208^{+}(0.2876)$	$0.2816^{***}(0.0900)$
δ_{Nov}	$0.4005^{**}(0.1651)$	$0.1798^{+}(0.1157)$
δ_{Dec}	$0.0780(0.1627)$	$0.1479(0.1326)$


$y_{1,t}$ World crude oil production growth
8
6
4
2
0
−2
−4
−6
−8
−10
1974
1980
1985
1991
1996
2002
2007
2013
2018


$y_{2,t}$ Industrial production growth of the US
3
2
1
0
−1
−2
−3
−4
−5
1974
1980
1985
1991
1996
2002
2007
2013
2018

Seasonal-QVAR: components

$y_t = c + \mu_t + s_t + v_t$

c is the vector of **constant parameters**.

μ_t is named as **local level**: dynamic interaction among the dependent variables in y_t.

s_t is the **stochastic seasonality** (time-varying amplitude of seasonality).

$v_t \sim t_K(0, \Sigma, \nu)$ is the multivariate i.i.d. **reduced-form error term** that updates y_t. $\Sigma = \Omega^{-1}(\Omega^{-1})'$ is positive definite and $\nu > 2$.

Seasonal-QVAR: components

$\mu_t = \Phi\mu_{t-1} + \Psi u_{t-1}$, where u_t $(K \times 1)$ is the scaled conditional score and Φ $(K \times K)$ and Ψ $(K \times K)$ are time constant parameter matrices.

The elements of s_t are modelled as $s_{k,t} = D_t' \rho_{k,t}$ for $k = 1, \ldots, K$,

where D_t $(S \times 1)$ is a vector of **seasonal dummy variables** and $\rho_{k,t}$ $(S \times 1)$ is a vector of **time-varying seasonal parameters**:

$\rho_{k,t} = \rho_{k,t-1} + \gamma_{k,t} u_{k,t-1}$

where $\gamma_{k,t}$ $(S \times 1)$ is a **time-varying vector of scaling parameters**:

Seasonal-QVAR: components

$\gamma_{k,t} = (\gamma_{k,1,t}, \dots, \gamma_{k,S,t})'$

The representative element $\gamma_{k,j,t}$ with $j = 1, \dots, S$ is:

$\gamma_{k,\mathbf{j},t} = \gamma_{k,j}$ if $D_{j,t} = 1$

$\gamma_{k,\mathbf{j},t} = -\gamma_{k,j}/(S-1)$ if $D_{j,t} = 0$

This specification ensures that $\sum_{j=1}^{S} \gamma_{k,j,t} = 0$, which implies that the seasonality component $s_{k,t}$ is centered at zero.

Seasonal-QVAR: scaled score function

The partial derivative of the log of the conditional density with respect to μ_t is

$$\frac{\partial \ln f(y_t | y_1, \ldots, y_{t-1})}{\partial \mu_t} = \frac{\nu + K}{\nu} \Sigma^{-1} \times \left(1 + \frac{v_t' \Sigma^{-1} v_t}{\nu}\right)^{-1} v_t =$$

$$= \frac{\nu + K}{\nu} \Sigma^{-1} \times u_t$$

where the last equality defines the **scaled score function** u_t.

Seasonal-QVAR: structural-form error term

We introduce the multivariate i.i.d. **structural-form error term** ε_t:

$$\varepsilon_t = v_t \left(\frac{\nu}{\nu-2}\right)^{-1/2} \Omega$$

where $E(\varepsilon_t) = 0_{K\times 1}$ and $\mathrm{Var}(\varepsilon_t) = I_K$.

The representation of the **score function** u_t by using the **structural-form error term** is

$$u_t = [(\nu-2)\nu]^{1/2} \Omega^{-1} \times \frac{\varepsilon_t}{\nu-2+\varepsilon_t'\varepsilon_t}$$

Seasonal-QVAR: Impulse response functions (IRFs)

The impulse response function $\mathrm{IRF}_{j,t} = \partial\mu_{t+j}/\partial\varepsilon_t$ for $j = 1, \dots, \infty$ is:

$\mathrm{IRF}_{j,t} = \Phi^j \Psi[(\nu - 2)\nu]^{1/2}\Omega^{-1}Q_{t-1-j}$ for $j = 1, \dots, \infty$,

where

$$Q_t = \frac{\partial \frac{\varepsilon_t}{\nu - 2 + \varepsilon_t' \varepsilon_t}}{\partial \varepsilon_t} \quad \rightarrow$$

$$\boldsymbol{IRF_j = \Phi^j \Psi[(\nu - 2)\nu]^{\frac{1}{2}} \Omega^{-1} E(Q_{t-1-j})}$$

Maximum likelihood (ML) estimation

The parameters of Seasonal-QVAR are estimated by using the ML method:

$$\hat{\Theta}_{\mathrm{ML}} = \arg\max_{\Theta} \mathrm{LL}(y_1, \dots, y_T) =$$

$$= \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t | y_1, \dots, y_{t-1})$$

We use the **inverse information matrix** to estimate the standard errors of parameters (Harvey 2013).

ML Condition 1

If all eigenvalues of Φ are within the unit circle, then y_t is covariance stationary.

Let C_1 denote the maximum modulus of all eigenvalues.

$C_1 < 1$ supports Condition 1.

ML Condition 2

We use Condition 2 from the work of Harvey (2013, p. 35, Condition 2).

Condition 2 is that the score function u_t $(K \times 1)$ and its first derivative $\partial u_t / \partial \mu_t$ $(K \times K)$ have finite second moments and covariance that are time-invariant and do not depend on μ_t.

ML Condition 3

Recall the first-order representation of QVAR:

$$\mu_t = \Phi\mu_{t-1} + \Psi u_{t-1}$$

$$\frac{\partial\mu_t}{\partial\Psi_{ij}} = \Phi\frac{\partial\mu_{t-1}}{\partial\Psi_{ij}} + \Psi\frac{\partial u_{t-1}}{\partial\Psi_{ij}} + W_{ij}u_{t-1}$$

$$\frac{\partial\mu_t}{\partial\Psi_{ij}} = \left(\Phi + \Psi\frac{\partial u_{t-1}}{\partial\mu'_{t-1}}\right)\frac{\partial\mu_{t-1}}{\partial\Psi_{ij}} + W_{ij}u_{t-1} = X_t\frac{\partial\mu_{t-1}}{\partial\Psi_{ij}} + W_{ij}u_{t-1}$$

Condition 3 is that all eigenvalues of $\mathrm{E}(X_t)$ are inside the unit circle (C_3 metric).

ML Condition 4

The information matrix for Seasonal-QVAR includes:

$$\mathrm{vec}\left(\frac{\partial\mu_t}{\partial\Psi_{ij}}\frac{\partial\mu_t'}{\partial\Psi_{kl}}\right) = (X_t \otimes X_t)\mathrm{vec}\left(\frac{\partial\mu_{t-1}}{\partial\Psi_{ij}}\frac{\partial\mu_{t-1}'}{\partial\Psi_{kl}}\right) +$$

$$\mathrm{vec}\left(X_t\frac{\partial\mu_{t-1}}{\partial\Psi_{ij}}W_{ij}'u_{t-1}\right) + \mathrm{vec}\left(u_{t-1}'W_{kl}\frac{\partial\mu_{t-1}'}{\partial\Psi_{kl}}X_t'\right) +$$

$$\mathrm{vec}\left(W_{ij}u_{t-1}u_{t-1}'W_{kl}'\right)$$

Condition 4 is that all eigenvalues of $E(X_t \otimes X_t)$ are inside the unit circle (C_4 metric: maximum modulus of all eigenvalues).

A note on the seasonality component

For the stochastic seasonality component s_t, the conditions of ML are satisfied when the parameter matrix corresponding to $\rho_{k,t-1}$ is set to the identity matrix as in equation $\rho_{k,t} = \rho_{k,t-1} + \gamma_{k,t} u_{k,t-1}$.

Classical alternatives

Gaussian Seasonal-VARMA: limiting special case

Seasonal-QVAR can be related to the Gaussian Seasonal-VARMA and Gaussian Seasonal-VAR models:

If $\nu \rightarrow \infty$, then $v_t \rightarrow N_K(0, \Sigma_v)$ and $u_t \rightarrow v_t$.

$$\left.\begin{aligned} y_t &= c + \mu_t + s_t + v_t \\ \mu_t &= \Phi\mu_{t-1} + \Psi v_{t-1} \end{aligned}\right\} \longrightarrow$$

$$\longrightarrow \quad y_t = (I_k - \Phi)c + \Phi y_{t-1} + (I_k - \Phi L)s_t + (\psi - \Phi)v_{t-1} + v_t$$

Seasonal-VARMA form

Basic structural model: local level

$y_t = c + \mu_t + s_t + v_t$

$\mu_t = \Phi\mu_{t-1} + \eta_t$ where $\eta_t \sim N_K(0_{K\times1}, \Sigma_\eta)$

The elements of s_t are modelled as $s_{k,t} = D_t'\rho_{k,t}$ for $k = 1, \dots, K$

$\rho_{k,t} = \rho_{k,t-1} + \xi_{k,t}$, where $\xi_{k,t} \sim N_S(0_{S\times1}, \Sigma_{\xi,k})$ i.i.d.

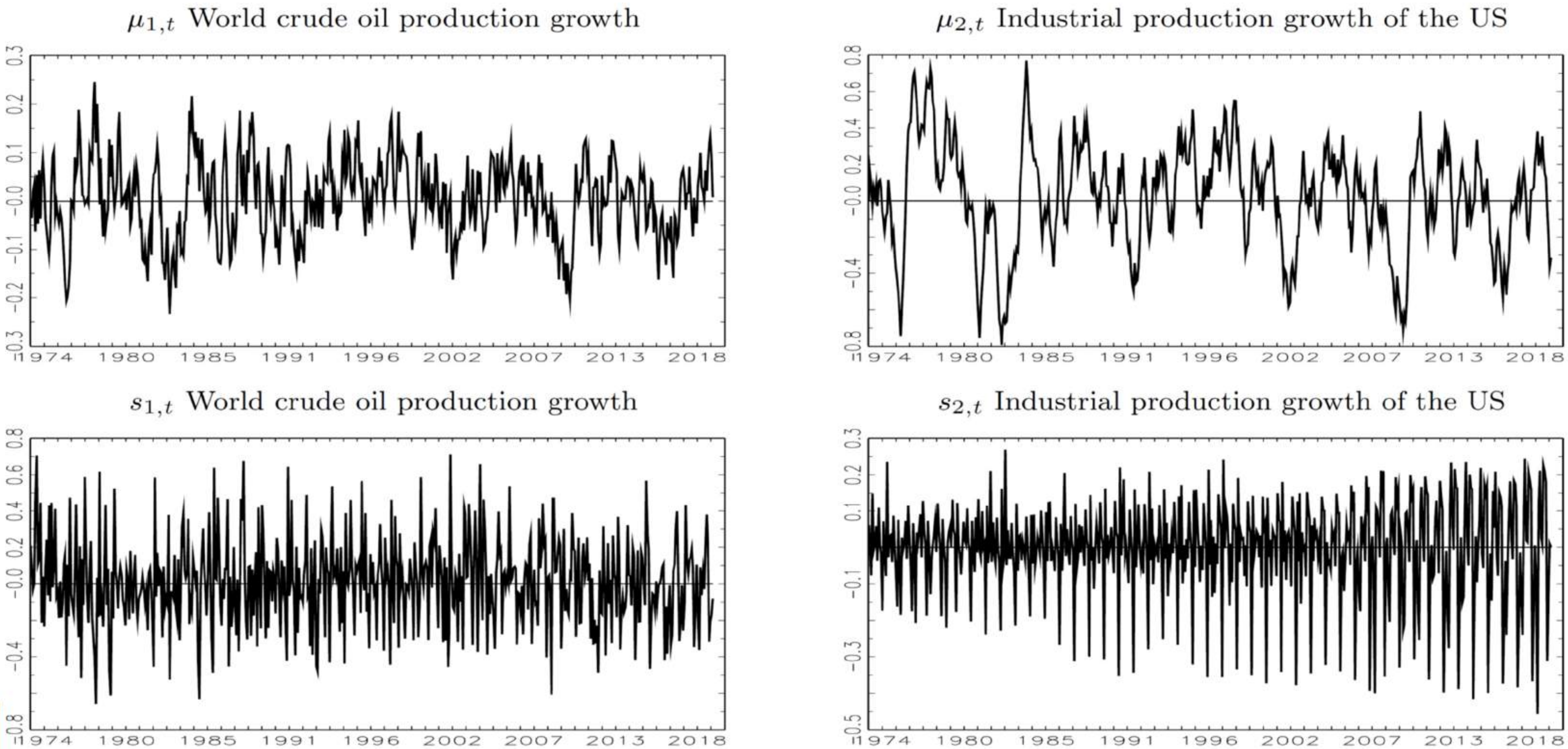
$\Sigma_{\xi,k} = \sigma_{\xi,k} \times (I_S - ii'/S)$, where $\sigma_{\xi,k} > 0$.

The structure of $\Sigma_{\xi,k}$ is such that all rows sum to zero and all columns sum to zero.

v_t is the i.i.d. Gaussian noise term (error term).

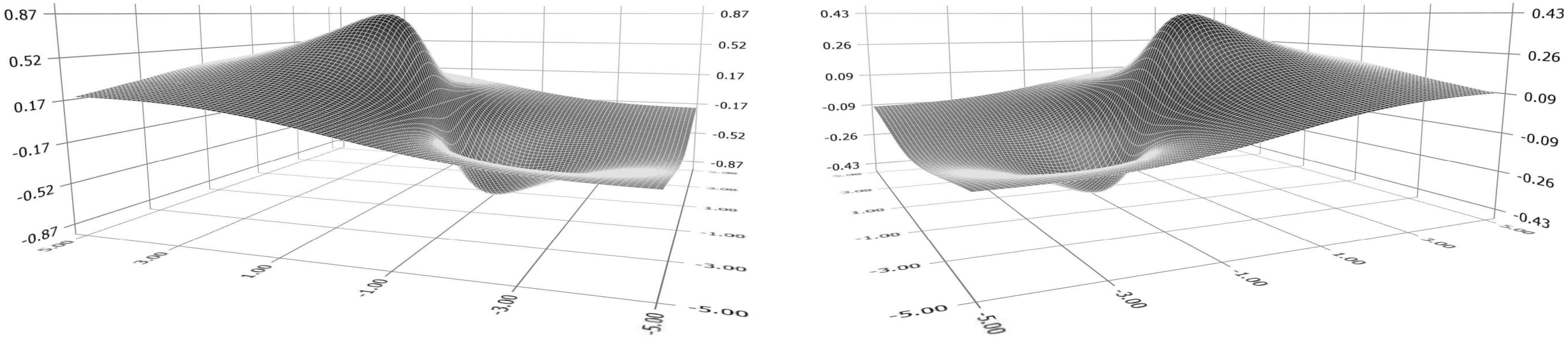
Seasonal-QVAR				Basic structural model		Gaussian Seasonal-VARMA			
c_1	$0.1416^{***}(0.0474)$	$\gamma_{2,\text{Jan}}$	$-0.1319(0.0995)$	c_1	$0.1010(0.2434)$	c_1	$0.0693^{**}(0.0338)$	$\gamma_{2,\text{Jan}}$	$0.0089(0.0280)$
c_2	$0.1274^{***}(0.0487)$	$\gamma_{2,\text{Feb}}$	$-0.1562^{**}(0.0714)$	c_2	$0.2073(0.2657)$	c_2	$0.1689^{***}(0.0387)$	$\gamma_{2,\text{Feb}}$	$-0.1500^{***}(0.0478)$
$\Phi_{1,1}$	$0.5257^{***}(0.1891)$	$\gamma_{2,\text{Mar}}$	$0.0416^{*}(0.0225)$	$\Phi_{1,1}$	$-0.2038^{*}(0.1107)$	$\Phi_{1,1}$	$0.5856^{***}(0.0703)$	$\gamma_{2,\text{Mar}}$	$0.1994^{*}(0.1033)$
$\Phi_{1,2}$	$0.1211(0.0871)$	$\gamma_{2,\text{Apr}}$	$0.2319^{***}(0.0737)$	$\Phi_{1,2}$	$0.3053^{+}(0.1914)$	$\Phi_{1,2}$	$0.2357^{***}(0.0730)$	$\gamma_{2,\text{Apr}}$	$-0.0536(0.0844)$
$\Phi_{2,1}$	$-0.4515(0.3779)$	$\gamma_{2,\text{May}}$	$-0.5553^{***}(0.2088)$	$\Phi_{2,1}$	$-0.2607^{+}(0.1766)$	$\Phi_{2,1}$	$-0.0769^{*}(0.0425)$	$\gamma_{2,\text{May}}$	$-0.0390(0.0600)$
$\Phi_{2,2}$	$0.9971^{***}(0.0933)$	$\gamma_{2,\text{Jun}}$	$0.0019(0.0402)$	$\Phi_{2,2}$	$0.8948^{***}(0.0870)$	$\Phi_{2,2}$	$0.8305^{***}(0.0428)$	$\gamma_{2,\text{Jun}}$	$0.0085(0.0197)$
$\Psi_{1,1}$	$-0.0869(0.0965)$	$\gamma_{2,\text{Jul}}$	$-0.1650(0.1538)$	$\Omega^{-1}_{v,1,1}$	$0.7632^{*}(0.4078)$	$\Psi_{1,1}$	$-0.2412^{***}(0.0709)$	$\gamma_{2,\text{Jul}}$	$-0.1032(0.1007)$
$\Psi_{1,2}$	$-0.0460(0.0883)$	$\gamma_{2,\text{Aug}}$	$0.0527(0.0499)$	$\Omega^{-1}_{v,2,1}$	$-0.4145^{***}(0.1147)$	$\Psi_{1,2}$	$0.0585(0.0507)$	$\gamma_{2,\text{Aug}}$	$0.0208(0.0400)$
$\Psi_{2,1}$	$0.0598^{+}(0.0381)$	$\gamma_{2,\text{Sep}}$	$-0.0696(0.0993)$	$\Omega^{-1}_{v,2,2}$	$0.0000(0.0005)$	$\Psi_{2,1}$	$0.0018(0.0162)$	$\gamma_{2,\text{Sep}}$	$-0.1410^{***}(0.0524)$
$\Psi_{2,2}$	$0.3746^{***}(0.0773)$	$\gamma_{2,\text{Oct}}$	$-0.0069(0.0460)$	$\Omega^{-1}_{\eta,1,1}$	$1.2032^{***}(0.2515)$	$\Psi_{2,2}$	$0.2827^{***}(0.0304)$	$\gamma_{2,\text{Oct}}$	$-0.0606(0.0408)$
$\Omega^{-1}_{v,1,1}$	$0.9242^{***}(0.0325)$	$\gamma_{2,\text{Nov}}$	$-0.0377(0.0691)$	$\Omega^{-1}_{\eta,2,1}$	$0.3591^{**}(0.1430)$	$\Omega^{-1}_{v,1,1}$	$1.4116^{***}(0.0366)$	$\gamma_{2,\text{Nov}}$	$0.0775(0.0566)$
$\Omega^{-1}_{v,2,1}$	$0.0552^{**}(0.0244)$	$\gamma_{2,\text{Dec}}$	$0.0935(0.1150)$	$\Omega^{-1}_{\eta,2,2}$	$0.2501^{***}(0.0348)$	$\Omega^{-1}_{v,2,1}$	$0.1013^{***}(0.0263)$	$\gamma_{2,\text{Dec}}$	$-0.0070(0.0179)$
$\Omega^{-1}_{v,2,2}$	$0.4549^{***}(0.0155)$	C_1	0.7909	$\sigma_{\xi,1}$	$0.0331^{***}(0.0094)$	$\Omega^{-1}_{v,2,2}$	$0.6286^{***}(0.0162)$	C_1	0.7103
ν	$3.5255^{***}(0.2899)$	C_2 to C_4 ADF	All stationary	$\sigma_{\xi,2}$	$0.0000(0.0037)$	$\gamma_{1,\text{Jan}}$	$-0.6723^{***}(0.1499)$	MDS $v_{1,t}$	0.6401
$\gamma_{1,\text{Jan}}$	$-0.0534(0.1814)$	C_3	0.7132	C_1	0.8168	$\gamma_{1,\text{Feb}}$	$-0.1083(0.0954)$	MDS $v_{2,t}$	0.4874
$\gamma_{1,\text{Feb}}$	$0.0728(0.2314)$	C_4	0.5146	MDS $v_{1,t}$	0.8817	$\gamma_{1,\text{Mar}}$	$0.2775^{***}(0.0976)$	LL	−2.7183
$\gamma_{1,\text{Mar}}$	$0.5342^{+}(0.3583)$	Q_t ADF	All stationary	MDS $v_{2,t}$	0.8909	$\gamma_{1,\text{Apr}}$	$0.3476^{***}(0.1285)$	AIC	5.5700
$\gamma_{1,\text{Apr}}$	$0.0833(0.2334)$	MDS $v_{1,t}$	0.2116	LL	−2.8375	$\gamma_{1,\text{May}}$	$0.2815^{***}(0.1024)$	BIC	5.8579
$\gamma_{1,\text{May}}$	$0.3969^{**}(0.1546)$	MDS $v_{2,t}$	0.0553	AIC	5.7254	$\gamma_{1,\text{Jun}}$	$0.4714^{***}(0.1204)$	HQC	5.6825
$\gamma_{1,\text{Jun}}$	$0.5691^{***}(0.1870)$	MDS $u_{1,t}$	0.2277	BIC	5.8343	$\gamma_{1,\text{Jul}}$	$-0.1628(0.1221)$		
$\gamma_{1,\text{Jul}}$	$-0.8191^{***}(0.1334)$	MDS $u_{2,t}$	0.1357	HQC	5.7679	$\gamma_{1,\text{Aug}}$	$0.3861^{***}(0.1173)$		
$\gamma_{1,\text{Aug}}$	$0.0701(0.1777)$	LL	**−2.5386**			$\gamma_{1,\text{Sep}}$	$-0.1182(0.1348)$		
$\gamma_{1,\text{Sep}}$	$-0.2525^{+}(0.1736)$	AIC	**5.2140**			$\gamma_{1,\text{Oct}}$	$0.1574(0.1098)$		
$\gamma_{1,\text{Oct}}$	$0.5815^{**}(0.2527)$	BIC	**5.5098**			$\gamma_{1,\text{Nov}}$	$0.0292(0.1084)$		
$\gamma_{1,\text{Nov}}$	$0.3040(0.2152)$	HQC	**5.3296**			$\gamma_{1,\text{Dec}}$	$-0.1717(0.1323)$		
$\gamma_{1,\text{Dec}}$	$-0.1211(0.2017)$								

Seasonal-QVAR: components



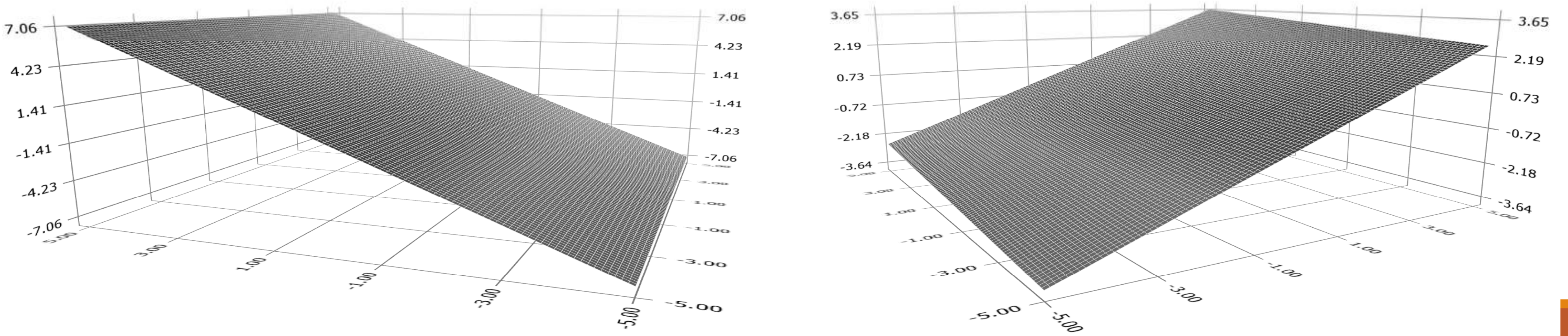
Seasonal-QVAR

$u_{1,t}$ as a function of $\epsilon_{1,t}$ and $\epsilon_{2,t}$ | $u_{2,t}$ as a function of $\epsilon_{1,t}$ and $\epsilon_{2,t}$



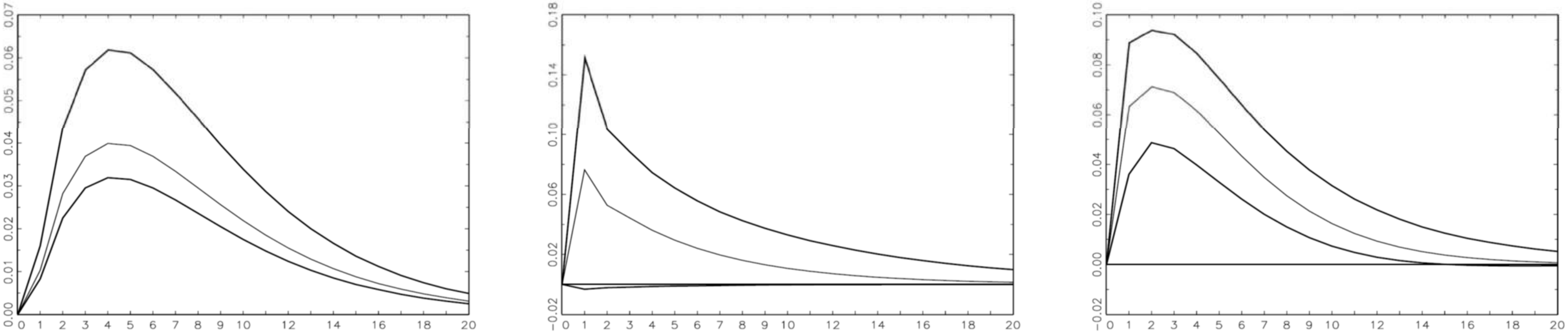
Seasonal-VARMA

$v_{1,t}$ as a function of $\epsilon_{1,t}$ and $\epsilon_{2,t}$ | $v_{2,t}$ as a function of $\epsilon_{1,t}$ and $\epsilon_{2,t}$



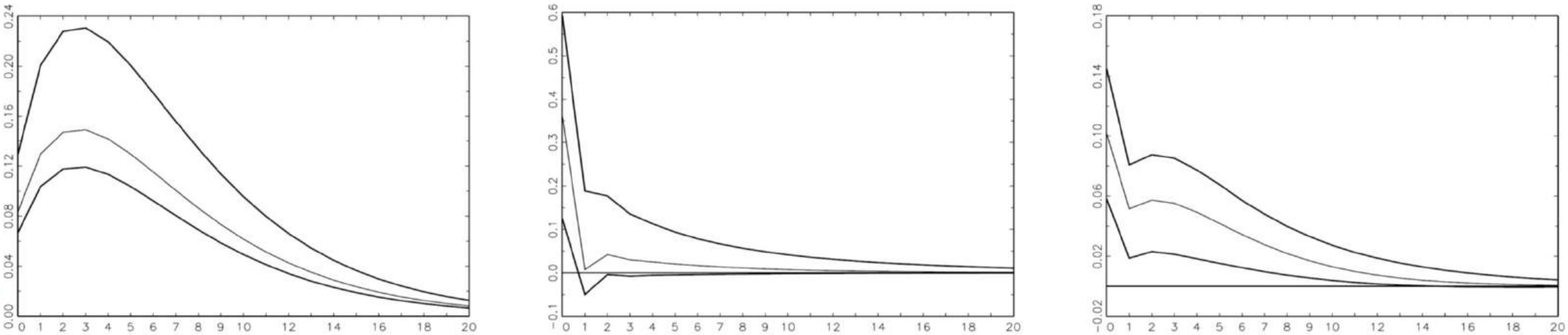
Industrial production growth of the US $\epsilon_{2,t} \longrightarrow$ Oil production growth $\mu_{1,t+j}$

Seasonal-QVAR | Basic structural model | Seasonal-VARMA



Oil production growth $\epsilon_{1,t} \longrightarrow$ Industrial production growth of the US $\mu_{2,t+j}$

Seasonal-QVAR | Basic structural model | Seasonal-VARMA



MS-Seasonal-QVAR

MS-Seasonal-QVAR: components

$y_t = c(r_t) + \mu_t(r_t) + s_t(r_t) + v_t(r_t)$

$\mu_t(r_t) = \Phi(r_t)\mu_{t-1}(r_t) + \Psi(r_t)u_{t-1}(r_t)$

$v_t \sim t_K[0, \Sigma(r_t), \nu(r_t)]$

The elements of $s_t(r_t)$are $s_{k,t}(r_t) = D_t'\rho_{k,t}(r_t)$for $k = 1, \dots, K$.

$\rho_{k,t}(r_t) = \rho_{k,t-1}(r_t) + \gamma_{k,t}(r_t)u_{k,t-1}(r_t)$

$\gamma_{k,j,t}(r_t) = \gamma_{k,j}(r_t)$if $D_{j,t} = 1$

$\gamma_{k,j,t}(r_t) = -\gamma_{k,j}(r_t)/(S-1)$ if $D_{j,t} = 0$

MS-Seasonal-QVAR: updating terms

Definitions of updating terms:

$$\mu_{t-1}(r_t) = E[\mu_{t-1}(r_{t-1})|y_1, \dots, y_{t-1}, r_t]$$

$$u_{t-1}(r_t) = E[u_{t-1}(r_{t-1})|y_1, \dots, y_{t-1}, r_t]$$

$$\rho_{k,t-1}(r_t) = E[\rho_{k,t-1}(r_{t-1})|y_1, \dots, y_{t-1}, r_t]$$

MS-Seasonal-QVAR: regimes

$r_t \in \{1,2\}$ for all t. Unobservable state variable (named regime).

r_t is a discrete-valued strictly stationary Markov process with transition probability matrix:

$$P = \begin{bmatrix} \Pr(r_t = 1 | r_{t-1} = 1) & \Pr(r_t = 2 | r_{t-1} = 1) \\ \Pr(r_t = 1 | r_{t-1} = 2) & \Pr(r_t = 2 | r_{t-1} = 2) \end{bmatrix} = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$$

$\Pr(r_t = 1) = \frac{1-q}{2-p-q} = \pi^*(1)$ (stationary probability)

$\Pr(r_t = 2) = 1 - \pi^*(1) = \pi^*(2)$ (stationary probability)

Maximum likelihood (ML) estimation

The parameters of Seasonal-QVAR are estimated by using the ML method:

$$\hat{\Theta}_{\mathrm{ML}} = \arg\max_{\Theta} \mathrm{LL}(y_1, \dots, y_T) =$$

$$= \arg\max_{\Theta} \sum_{t=1}^{T} \ln f(y_t | y_1, \dots, y_{t-1})$$

We use the **inverse information matrix** to estimate the standard errors of parameters (Harvey 2013).

ML Condition 1

$$
\begin{aligned}
\mu_t(r_t) &= \Phi(r_t)\mu_{t-1}(r_t) + \Psi(r_t)u_{t-1}(r_t) \\
&= \Phi(r_t)E[\mu_{t-1}(r_{t-1})|y_1, \ldots, y_{t-1}, r_t] + \Psi(r_t)E[u_{t-1}(r_{t-1})|y_1, \ldots, y_{t-1}, r_t] \\
&= \Phi(r_t)\mu_{t-1}(r_{t-1} = 1)\Pr(r_{t-1} = 1|y_1, \ldots, y_{t-1}, r_t) \\
&\quad +\Phi(r_t)\mu_{t-1}(r_{t-1} = 2)\Pr(r_{t-1} = 2|y_1, \ldots, y_{t-1}, r_t) \\
&\quad +\Psi(r_t)u_{t-1}(r_{t-1} = 1)\Pr(r_{t-1} = 1|y_1, \ldots, y_{t-1}, r_t) \\
&\quad +\Psi(r_t)u_{t-1}(r_{t-1} = 2)\Pr(r_{t-1} = 2|y_1, \ldots, y_{t-1}, r_t)
\end{aligned}
$$

ML Condition 1

$$E[\mu_t(r_t)|y_1,\ldots,y_{t-1},r_t] = \Phi(r_t)\Pr(r_{t-1}|r_t)E[\mu_{t-1}(r_{t-1})|y_1,\ldots,y_{t-2},r_{t-1}]$$
$$+\Psi(r_t)\Pr(r_{t-1}|r_t)E[u_{t-1}(r_{t-1})|y_1,\ldots,y_{t-2},r_{t-1}]$$

→ recursive equation.

ML Condition 1

The probabilities in the previous equation are:

$$\Pr(r_{t-1} = 1 | r_t = 1) = \Pr(r_t = 1 | r_{t-1} = 1) = p$$

$$\Pr(r_{t-1} = 2 | r_t = 1) = \frac{\pi^*(2)}{\pi^*(1)} \Pr(r_t = 1 | r_{t-1} = 2) = \frac{\pi^*(2)}{\pi^*(1)} (1 - q)$$

$$\Pr(r_{t-1} = 1 | r_t = 2) = \frac{\pi^*(1)}{\pi^*(2)} \Pr(r_t = 2 | r_{t-1} = 1) = \frac{\pi^*(1)}{\pi^*(2)} (1 - p)$$

$$\Pr(r_{t-1} = 2 | r_t = 2) = \Pr(r_t = 2 | r_{t-1} = 2) = q$$

ML Condition 1

By considering all possible values for r_t and r_{t-1} in the recursive equation

$$E[\mu_t(r_t)|y_1,\ldots,y_{t-1},r_t] = \Phi(r_t)\Pr(r_{t-1}|r_t)E[\mu_{t-1}(r_{t-1})|y_1,\ldots,y_{t-2},r_{t-1}] + \Psi(r_t)\Pr(r_{t-1}|r_t)E[u_{t-1}(r_{t-1})|y_1,\ldots,y_{t-2},r_{t-1}]$$

and substituting the conditional probability formulas, we define the following matrix:

ML Condition 1

$$\begin{bmatrix} \Phi(1)p & \Phi(1)\frac{\pi^*(2)}{\pi^*(1)}(1-q) \\ \Phi(2)\frac{\pi^*(1)}{\pi^*(2)}(1-p) & \Phi(2)q \end{bmatrix}$$

C1 (condition of covariance stationarity of μ_t): The maximum modulus of all eigenvalues of this matrix is less than one.

ML Condition 2

Technical condition:

$$E[u_{j,t-1}^{2-i}(r_t)[\partial u_{k,t-1}(r_t)/\partial \mu_{l,t-1}(r_t)]^i] \ < \ \infty$$

for $i = 0,1,2; j, k, l = 1, \dots, K; r_t = 1,2.$

ML Condition 3

$$\frac{\partial \mu_t(r_t)}{\partial \Psi_{i,j}(r_t)} = \left[\Phi(r_t) + \Psi(r_t)\frac{\partial u_{t-1}(r_t)}{\partial \mu'_{t-1}(r_t)}\right]\frac{\partial \mu_{t-1}(r_t)}{\partial \Psi_{i,j}(r_t)} + W_{i,j}u_{t-1}(r_t)$$

$$= X_t(r_t)\frac{\partial \mu_{t-1}(r_t)}{\partial \Psi_{i,j}(r_t)} + W_{i,j}u_{t-1}(r_t)$$

By using arguments that are similar to the arguments for (C1), we define the following matrix:

ML Condition 3

$$\begin{Bmatrix} E[X_t(1)]p & E[X_t(1)]\frac{\pi^*(2)}{\pi^*(1)}(1-q) \\ E[X_t(2)]\frac{\pi^*(1)}{\pi^*(2)}(1-p) & E[X_t(2)]q \end{Bmatrix}$$

C3: The maximum modulus of all eigenvalues of this matrix is less than one.

ML Condition 4

$$\text{vec}\left[\frac{\partial\mu_t(r_t)}{\partial\Psi_{i,j}(r_t)}\frac{\partial\mu_t'(r_t)}{\partial\Psi_{k,l}(r_t)}\right] = [X_t(r_t)\otimes X_t(r_t)]\text{vec}\left[\frac{\partial\mu_{t-1}(r_t)}{\partial\Psi_{i,j}(r_t)}\frac{\partial\mu_{t-1}'(r_t)}{\partial\Psi_{k,l}(r_t)}\right]$$

$$+\text{vec}\left[X_t(r_t)\frac{\partial\mu_{t-1}(r_t)}{\partial\Psi_{i,j}(r_t)}W_{i,j}'u_{t-1}(r_t)\right] + \text{vec}\left[u_{t-1}'(r_t)W_{k,l}\frac{\partial\mu_{t-1}'(r_t)}{\partial\Psi_{k,l}(r_t)}X_t'(r_t)\right]$$

$$+\text{vec}\left[W_{i,j}u_{t-1}(r_t)u_{t-1}'(r_t)W_{k,l}'\right]$$

ML Condition 4

By using arguments that are similar to the arguments for (C1) and (C3), we define the following matrix:

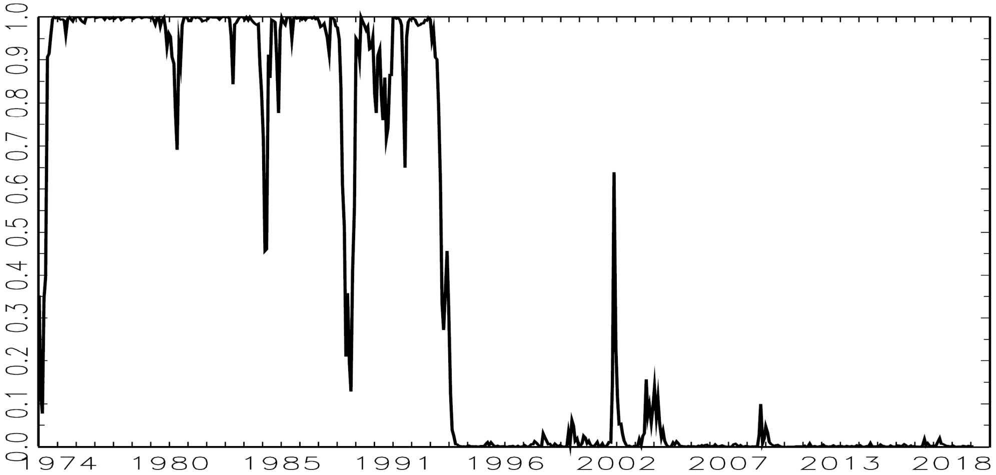
$$\begin{Bmatrix} E[X_t(1) \otimes X_t(1)]p & E[X_t(1) \otimes X_t(1)]\frac{\pi^*(2)}{\pi^*(1)}(1-q) \\ E[X_t(2) \otimes X_t(2)]\frac{\pi^*(1)}{\pi^*(2)}(1-p) & E[X_t(2) \otimes X_t(2)]q \end{Bmatrix}$$

C4: The maximum modulus of all eigenvalues of this matrix is less than one.

MS-Seasonal-QVAR, specification: $c(r_t)$, $\mu_t(r_t)$, s_t, $v_t(r_t)$				MS-Seasonal-QVAR, specification: $c(r_t)$, μ_t, s_t, $v_t(r_t)$			
p	0.9976***(0.0002)	$\gamma_{1,Jan}$	0.1895***(0.0340)	p	0.9976***(0.0028)	$\gamma_{1,Jan}$	−0.0799(0.1696)
q	0.9981***(0.0001)	$\gamma_{1,Feb}$	0.1211***(0.0437)	q	0.9982***(0.0019)	$\gamma_{1,Feb}$	0.2034(0.2088)
$c_1(1)$	0.2687***(0.0315)	$\gamma_{1,Mar}$	0.3029*(0.1549)	$c_1(1)$	0.2519**(0.1073)	$\gamma_{1,Mar}$	0.3188(0.2282)
$c_2(1)$	0.1403***(0.0246)	$\gamma_{1,Apr}$	−0.0450(0.0314)	$c_2(1)$	0.1768**(0.0830)	$\gamma_{1,Apr}$	0.1268(0.2493)
$c_1(2)$	0.1292***(0.0269)	$\gamma_{1,May}$	0.3354***(0.0598)	$c_1(2)$	0.0940**(0.0384)	$\gamma_{1,May}$	0.1894(0.2143)
$c_2(2)$	0.2195***(0.0287)	$\gamma_{1,Jun}$	0.1286**(0.0553)	$c_2(2)$	0.1046*(0.0578)	$\gamma_{1,Jun}$	0.5371**(0.2220)
$\Phi_{1,1}(1)$	−0.1087**(0.0530)	$\gamma_{1,Jul}$	−0.8161***(0.0462)	$\Phi_{1,1}$	0.5368***(0.1056)	$\gamma_{1,Jul}$	−0.4680**(0.2003)
$\Phi_{1,2}(1)$	0.2821***(0.0205)	$\gamma_{1,Aug}$	−0.0870***(0.0334)	$\Phi_{1,2}$	0.1678**(0.0727)	$\gamma_{1,Aug}$	0.0529(0.2364)
$\Phi_{2,1}(1)$	−2.8013***(0.0841)	$\gamma_{1,Sep}$	−0.3947***(0.0289)	$\Phi_{2,1}$	−0.2270*(0.1335)	$\gamma_{1,Sep}$	−0.2137(0.1819)
$\Phi_{2,2}(1)$	1.6327***(0.0445)	$\gamma_{1,Oct}$	0.4007***(0.0346)	$\Phi_{2,2}$	0.9192***(0.0550)	$\gamma_{1,Oct}$	0.6351**(0.2563)
$\Phi_{1,1}(2)$	0.8648***(0.0602)	$\gamma_{1,Nov}$	−0.0550(0.0403)			$\gamma_{1,Nov}$	0.2174(0.2402)
$\Phi_{1,2}(2)$	−0.0326**(0.0150)	$\gamma_{1,Dec}$	−0.0036(0.0494)			$\gamma_{1,Dec}$	−0.0412(0.1898)
$\Phi_{2,1}(2)$	0.5612***(0.1136)	$\gamma_{2,Jan}$	−0.8408***(0.0631)			$\gamma_{2,Jan}$	0.0196(0.0285)
$\Phi_{2,2}(2)$	0.8800***(0.0442)	$\gamma_{2,Feb}$	0.2514***(0.0600)			$\gamma_{2,Feb}$	0.0074(0.0280)
$\Psi_{1,1}(1)$	−0.0295**(0.0129)	$\gamma_{2,Mar}$	−0.0198(0.0170)	$\Psi_{1,1}$	−0.2434*(0.1326)	$\gamma_{2,Mar}$	0.2043(0.1467)
$\Psi_{1,2}(1)$	0.2543***(0.0345)	$\gamma_{2,Apr}$	0.1588***(0.0366)	$\Psi_{1,2}$	0.0072(0.0783)	$\gamma_{2,Apr}$	−0.0710(0.0999)
$\Psi_{2,1}(1)$	0.0206(0.0311)	$\gamma_{2,May}$	−0.4214***(0.0574)	$\Psi_{2,1}$	−0.0021(0.0301)	$\gamma_{2,May}$	−0.0786(0.0745)
$\Psi_{2,2}(1)$	0.7459***(0.0876)	$\gamma_{2,Jun}$	0.0321(0.0348)	$\Psi_{2,2}$	0.3310***(0.0622)	$\gamma_{2,Jun}$	0.0262(0.0231)
$\Psi_{1,1}(2)$	−0.1000***(0.0366)	$\gamma_{2,Jul}$	−0.1295***(0.0423)			$\gamma_{2,Jul}$	−0.2667*(0.1480)
$\Psi_{1,2}(2)$	0.1563***(0.0411)	$\gamma_{2,Aug}$	0.1624***(0.0342)			$\gamma_{2,Aug}$	0.0132(0.0302)
$\Psi_{2,1}(2)$	0.0377(0.0315)	$\gamma_{2,Sep}$	−0.3978***(0.0717)			$\gamma_{2,Sep}$	−0.0036(0.0456)
$\Psi_{2,2}(2)$	0.1469***(0.0409)	$\gamma_{2,Oct}$	0.3433***(0.0691)			$\gamma_{2,Oct}$	0.0506*(0.0267)
$\Omega^{-1}_{v,1,1}(1)$	1.4990***(0.0571)	$\gamma_{2,Nov}$	−0.0194(0.0176)	$\Omega^{-1}_{v,1,1}(1)$	1.6021***(0.1056)	$\gamma_{2,Nov}$	−0.0412*(0.0222)
$\Omega^{-1}_{v,2,1}(1)$	0.0999***(0.0277)	$\gamma_{2,Dec}$	0.2333***(0.0434)	$\Omega^{-1}_{v,2,1}(1)$	0.1208***(0.0375)	$\gamma_{2,Dec}$	0.1432(0.1068)
$\Omega^{-1}_{v,2,2}(1)$	0.4935***(0.0159)			$\Omega^{-1}_{v,2,2}(1)$	0.5244***(0.0330)		
$\Omega^{-1}_{v,1,1}(2)$	0.6474***(0.0189)			$\Omega^{-1}_{v,1,1}(2)$	0.6494***(0.0310)		
$\Omega^{-1}_{v,2,1}(2)$	0.0338(0.0273)			$\Omega^{-1}_{v,2,1}(2)$	0.0335(0.0278)		
$\Omega^{-1}_{v,2,2}(2)$	0.4396***(0.0154)			$\Omega^{-1}_{v,2,2}(2)$	0.4826***(0.0242)		
$\nu(1)$	3.9071***(0.3172)			$\nu(1)$	4.3035***(0.7450)		
$\nu(2)$	6.8669***(0.6030)			$\nu(2)$	7.9438***(1.8892)		

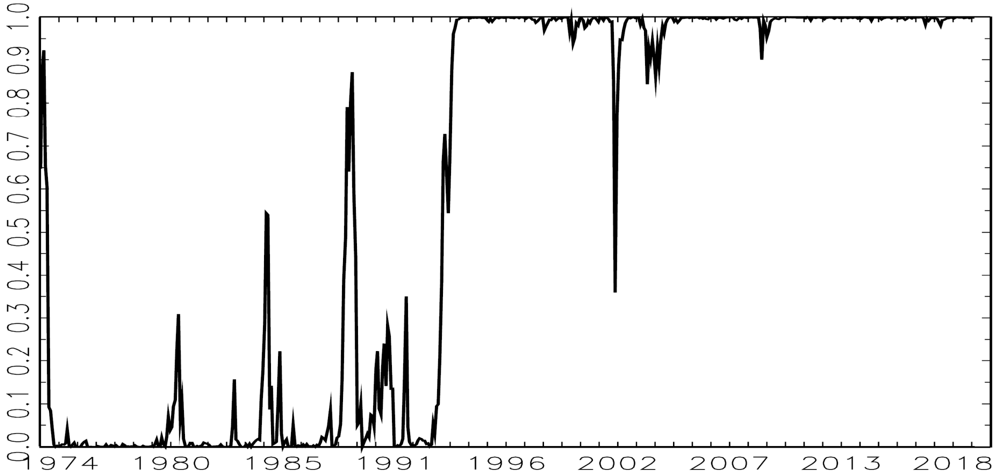
High-volatility regime ($r_t = 1$)

Filtered probability: $\tilde{\pi}_t(1) = \Pr(r_1 = 1 | y_1, \ldots, y_t)$



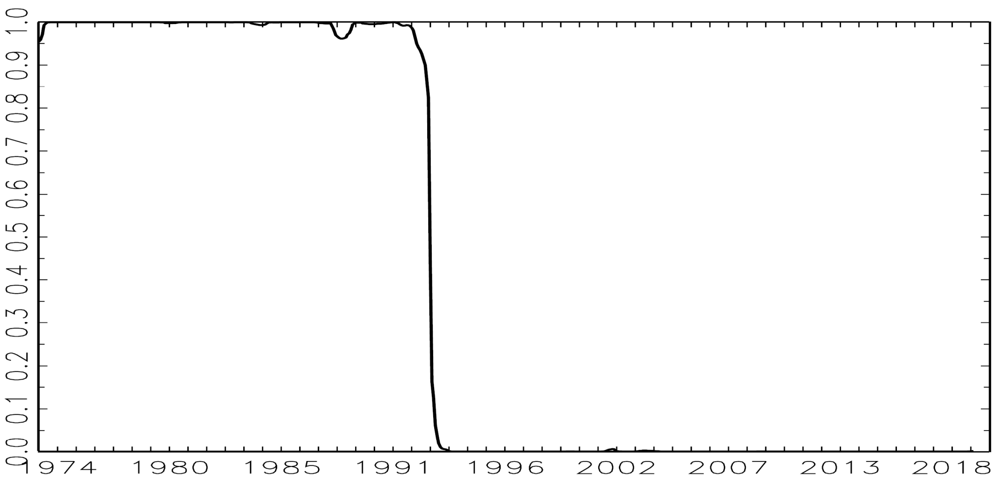
Low-volatility regime ($r_t = 2$)

Filtered probability: $\tilde{\pi}_t(2) = \Pr(r_1 = 2 | y_1, \ldots, y_t)$



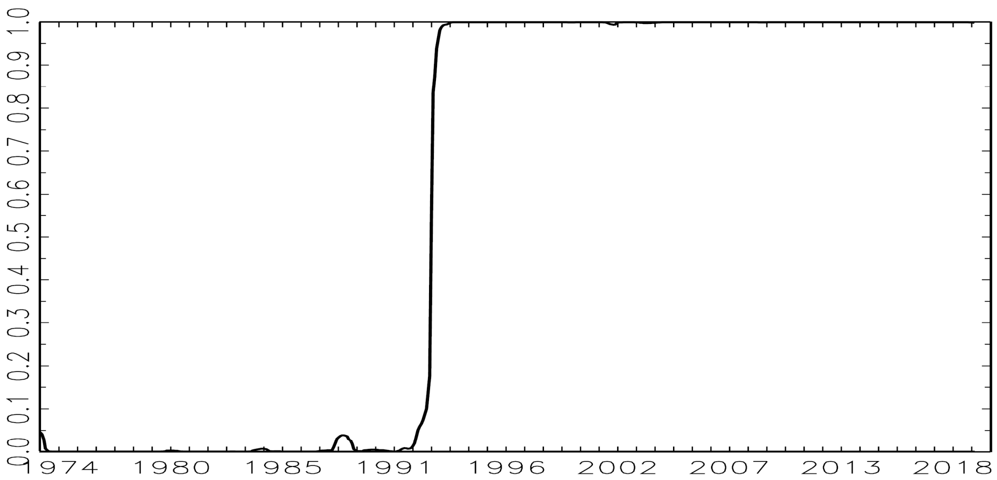
High-volatility regime ($r_t = 1$)

Smoothed probability: $\Pr(r_1 = 1 | y_1, \ldots, y_T)$



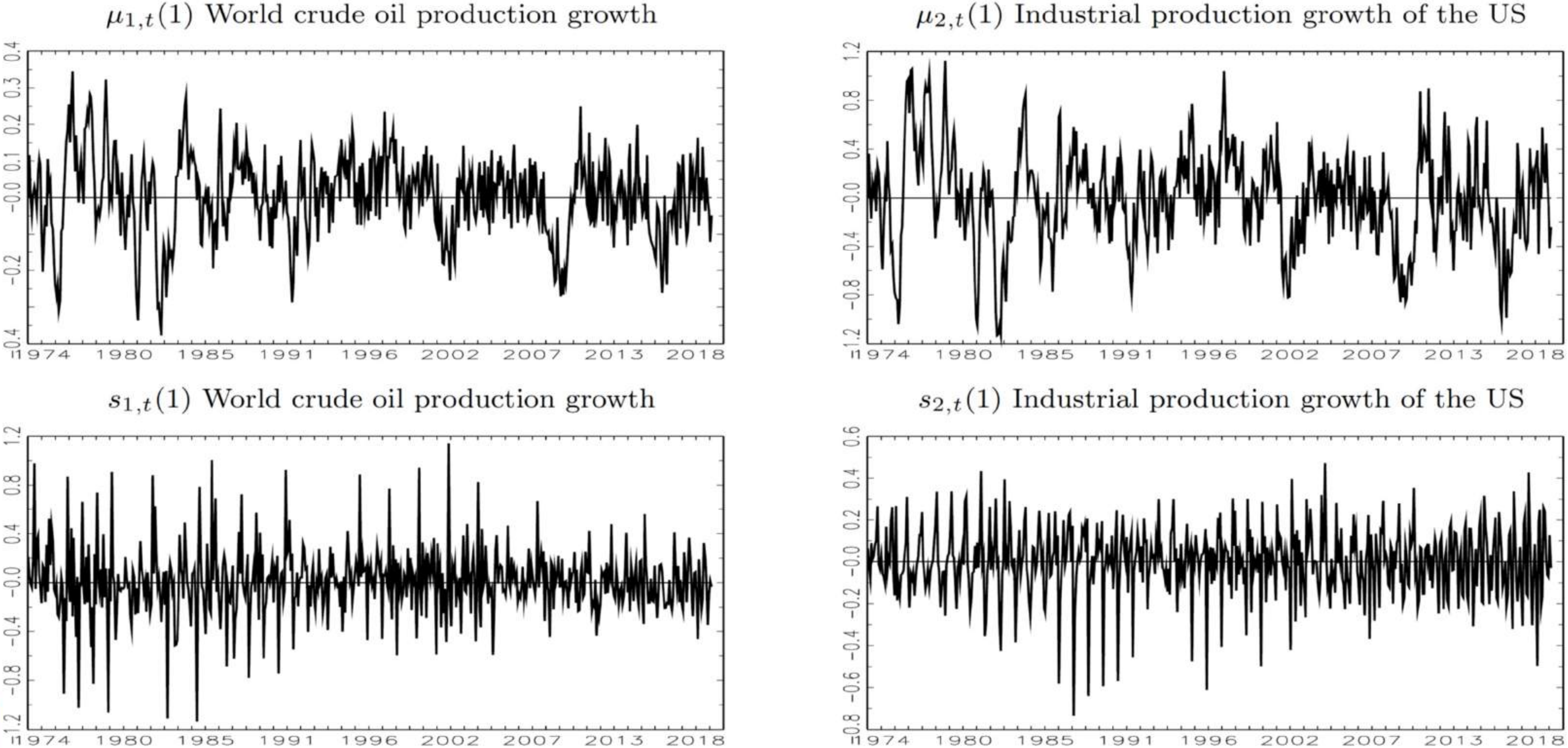
Low-volatility regime ($r_t = 2$)

Smoothed probability: $\Pr(r_1 = 2 | y_1, \ldots, y_T)$

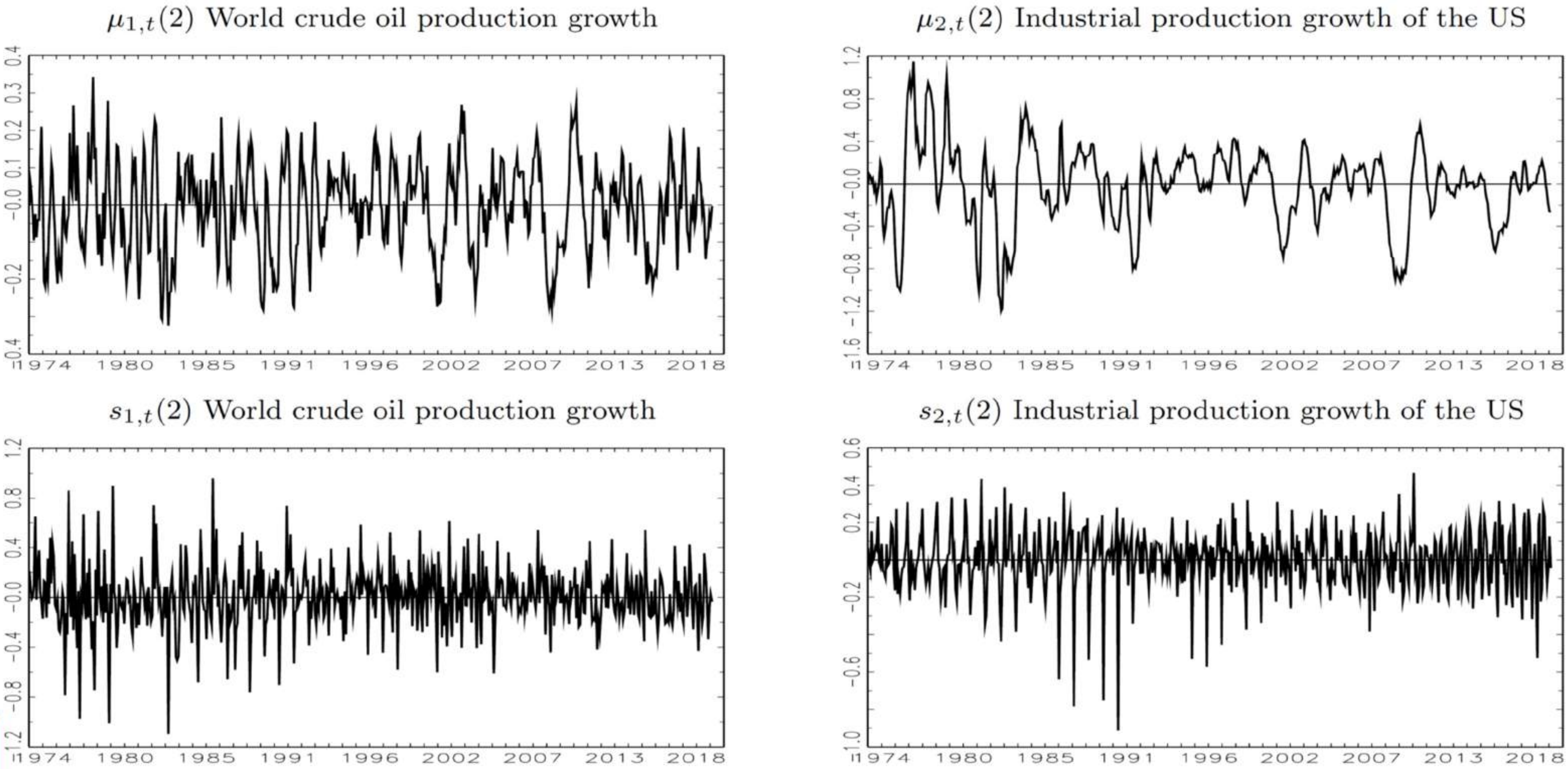


MS-Seasonal-QVAR, specification: $c(r_t)$, $\mu_t(r_t)$, s_t, $v_t(r_t)$		MS-Seasonal-QVAR, specification: $c(r_t)$, μ_t, s_t, $v_t(r_t)$	
$C_1(1)$	0.7828	$C_1(1)$	0.7291
$C_1(2)$	0.8828	$C_1(2)$	0.7291
$Q_t(1)$ ADF	All stationary	$Q_t(1)$ ADF	All stationary
$Q_t(2)$ ADF	All stationary	$Q_t(2)$ ADF	All stationary
C_1	0.8810	C_1	0.7291
C_2 to C_4 ADF	All stationary	C_2 to C_4 ADF	All stationary
C_3	0.8773	C_3	0.7321
C_4	0.7771	C_4	0.5443
MDS $E(v_{1,t})$	0.2750	MDS $E(v_{1,t})$	0.4067
MDS $E(v_{2,t})$	0.5628	MDS $E(v_{2,t})$	0.1802
MDS $E(u_{1,t})$	0.4915	MDS $E(u_{1,t})$	0.3310
MDS $E(u_{2,t})$	0.2410	MDS $E(u_{2,t})$	0.3062
LL	**−2.3719**	LL	−2.4098
AIC	**4.9384**	AIC	4.9853
BIC	5.3586	BIC	**5.3433**
HQC	**5.1025**	HQC	5.1252
LR	**42.0726*****		

Components: high-volatility regime

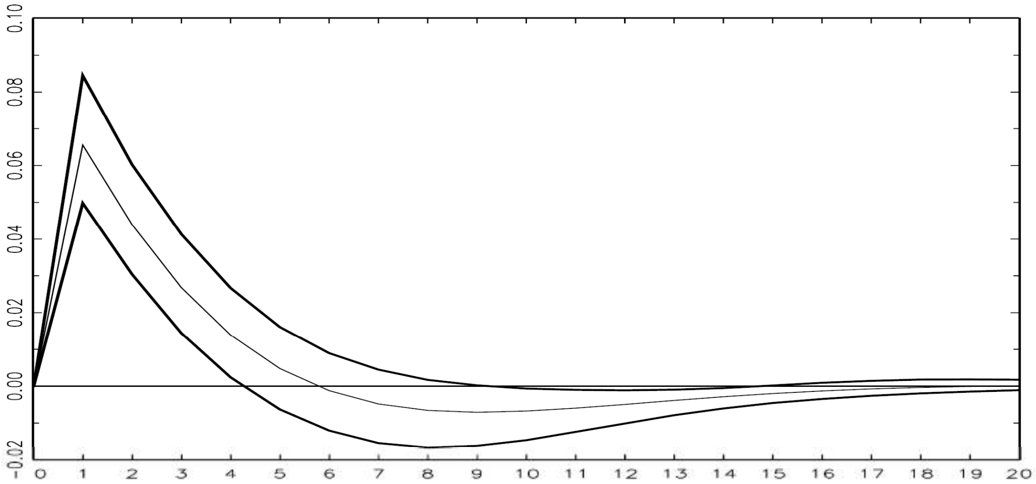


Components: low-volatility regime

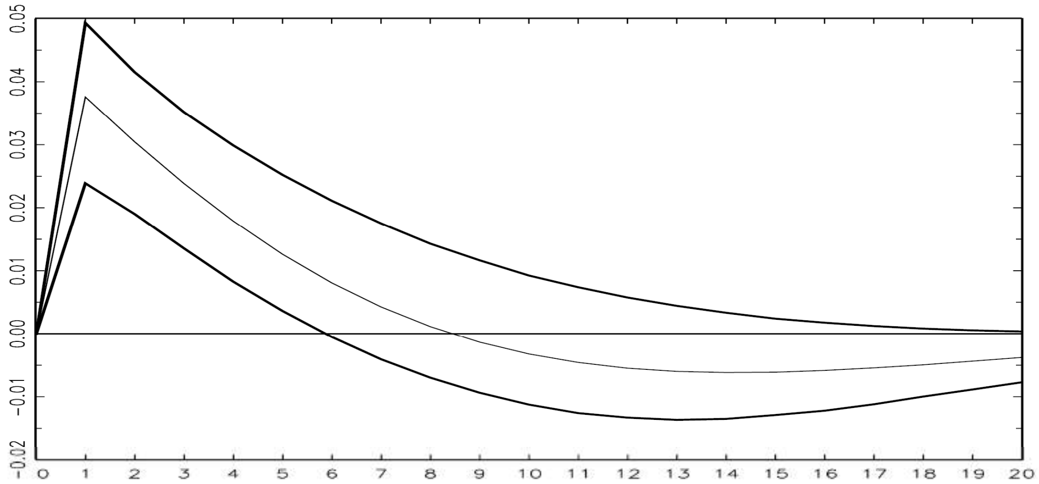


Industrial production growth of the US $\epsilon_{2,t} \longrightarrow$ Oil production growth $\mu_{1,t+j}$

High-volatility regime ($r_t = 1$)

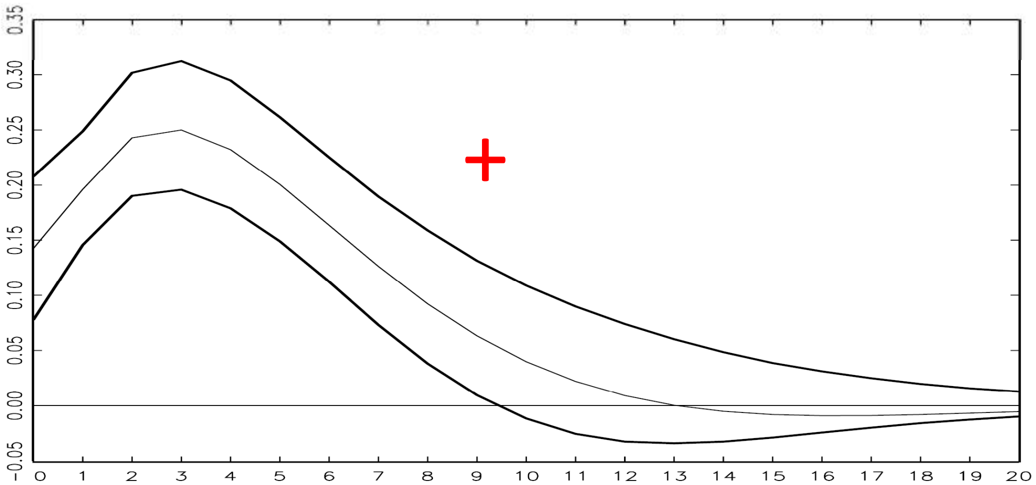


Low-volatility regime ($r_t = 2$)

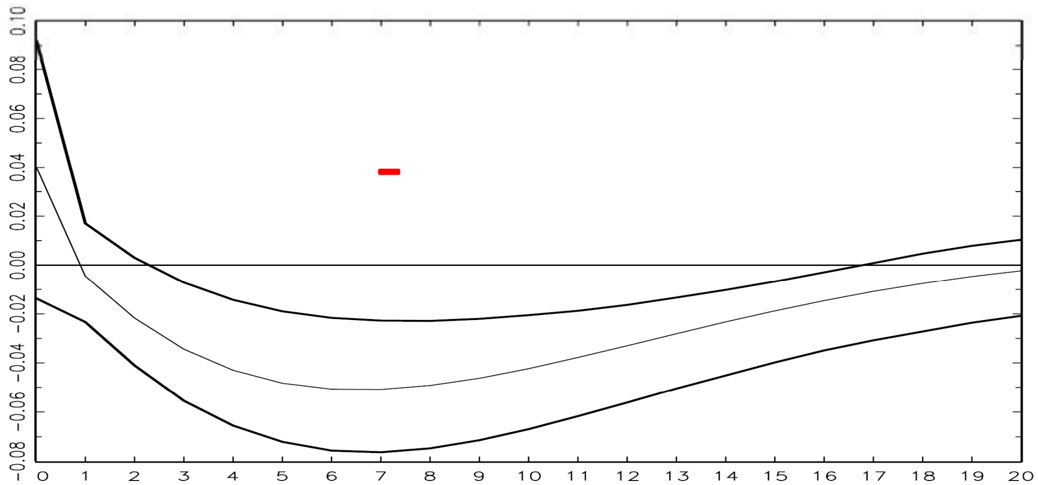


Oil production growth $\epsilon_{1,t} \longrightarrow$ Industrial production growth of the US $\mu_{2,t+j}$

High-volatility regime ($r_t = 1$)



Low-volatility regime ($r_t = 2$)



Conclusions

IRF results in relation to: Baumeister and Peersman (2013); Baumeister and Hamilton (2017); Kilian and Lütkepohl (2017).

Statistical performance of MS-Seasonal-QVAR is superior to Seasonal-QVAR and Gaussian alternatives.

Possible alternative: Score-driven versions of vector threshold autoregressive (VTAR) and vector smooth transition autoregressive (VSTAR) models (motivated by the smoothed probability estimates).

Thank you for your attention

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